Norwegian University of Science
and Technology
Institute of Mathematical Sciences

Exercise set 1

1 **a)** Apply Newton's method to the equation $f(x) = 0$, where

i) $f(x) = \cos x - 1/2$, with $x_0 = 0.5$.

ii) $f(x) = e^x - x - 1$, with $x_0 = 0.5$.

iii) $f(x) = x(1 - \cos x)$, with $x_0 = 0.5$.

Use Python/Matlab or any other programming language:
The iterations will converge to a root, $x^\star$, of $f(x)$ in all three cases. Measure the order of convergence using e.g. Python/Matlab in the three cases. Is the result in accordance with theory? If no, can you explain why?
Note: For some of the equations you may encounter problems with machine precision. Relative errors of approximately the same magnitude as machine epsilon ($\approx 2.22 \cdot 10^{-16}$) are usually not due to the numerical method.

We say that a root, $x^\star$, of $f(x)$ has multiplicity m if there exists a function $q(x)$ such that

$$f(x) = (x - x^\star)^m q(x), \qquad q(x^\star) \neq 0,$$

which is the case if and only if

$$f(x^\star) = f'(x^\star) = \cdots = f^{(m-1)}(x^\star) = 0, \qquad f^{(m)}(x^\star) \neq 0.$$

b) What is the multiplicity of the solutions of the three equations in **a)**?

c) Assume that $x^\star$ is a root with multiplicity m of the function $f(x)$. Show that the function

$$\mu(x) = f(x)/f'(x)$$

has a simple root in $x^\star$, independent of m. Use this to find an iteration scheme that converges quadratically to $x^\star$.

d) Test the new scheme on the functions *ii)* and *iii)* in **a)**.

e) Repeat the task in **a)** using the secant method instead of Newton's method.

2 Consider the system of equations

$$x_1^2 + x_2^2 = 1,$$
$$x_1^3 - x_2 = 0.$$

This has two solutions, one in the region $-1 \leq x_1, x_2 \leq 0$ and one in $0 \leq x_1, x_2 \leq 1$.

a) Choose appropriate initial values and perform two iterations by hand using Newton's method.

b) Verify that you get correct answers using Python/Matlab.

c) Explain what happens when you choose initial value lying on the x_2-axis.

3 Consider the sequence

$$x_k = 2^{-k^\alpha}, \qquad k = 1, 2, 3, \ldots$$

where $\alpha > 0$. It is easily seen that

$$\lim_{k \to \infty} x_k = 0$$

Use Definition 1.4 in S&M and discuss the convergence of (x_k) for different α. When does the sequence converge linearly? Superlinearly? Sublinearly?

Extra: Use Definition 1.7 in S&M. Does the sequence converge with order $q > 1$ for any α?

Note: S&M,

Süli, Endre, and David F. Mayers. An introduction to numerical analysis. Cambridge university press, 2003.