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Problem 1

a) Find the polynomial $p \in \Pi_3$, such that

$$p(0) = 1, \quad p(1) = -1, \quad p'(0) = 1, \quad p''(0) = 0.$$

Solution: the polynomial is

$$p(x) = -3x^3 + x + 1.$$

Problem 2

a) Let $n \geq 2$. Show that a polynomial p_{2n-1} of degree $2n-1$ can be written

$$p_{2n-1}(x) = (x-a)(b-x)q_{2n-3}(x) + r(x-a) + s(b-x),$$

where q_{2n-3} is a polynomial of degree $2n-3$, and a, b, r and s are constants.

Hint. Observe that the set of polynomials

$$(x-a), \quad (b-x), \quad (x-a)\cdot(b-x)\cdot x^k, \quad k = 0, \dots, 2n-3,$$

with a and b not simultaneously equal to zero, is a basis for the vector space of polynomials of degree less than or equal to $2n-1$.

Solution: Using the given basis any polynomial $p_{2n-1}(x)$ can be written in the form

$$p_{2n-1}(x) = b_0(x-a) + b_1(b-x) + \sum_{k=0}^{2n-3} b_k \left((x-a)\cdot(b-x)\cdot x^k\right),$$

so the right hand side of the equality we have to show is obtained for $r = b_0$, $s = b_1$ and

$$q_{2n-3}(x) = \sum_{k=0}^{2n-3} b_k\, x^k,$$

a polynomial of degree less than or equal to $2n-3$.

b) Then construct the Lobatto quadrature formula

$$\int_a^b w(x)f(x)\,dx \approx W_0 f(a) + \sum_{k=1}^{n-1} W_k f(x_k) + W_n f(b),$$

which is exact when f is a polynomial of degree $2n-1$. Here $w(x)$ is a positive weight function.

Hint. One way to solve this problem is by using the $n-1$ Gauss quadrature points and weights with respect to a weight function $\tilde{w}(x) \neq w(x)$.

Solution. The Lobatto quadrature formulae have the form

$$W_0 f(a) + \sum_{k=1}^{n-1} W_k f(x_k) + W_n f(b),$$

and are exact for all polynomials $p_{2n-1}(x)$ of degree less than or equal to $2n-1$. We use the expression for p_{2n-1} obtained in the previous exercise. Observe that $p_{2n-1}(a) = s(b-a)$ and $p_{2n-1}(b) = r(b-a)$, then

$$\int_a^b w(x)p_{2n-1}(x)\,dx = \int_a^b w(x)(x-a)(b-x)q_{2n-3}(x)\,dx + r(b-a)W_0 + s(b-a)W_n$$

where $W_0 := (b-a)^{-1}\int_a^b w(x)(x-a)\,dx$ and $W_n := (b-a)^{-1}\int_a^b w(x)(b-x)\,dx$. We now choose $x_1, \dots, x_{n-1}$ and $W_1, \dots, W_{n-1}$ to be the $n-1$ Gauss quadrature nodes and weights with respect to the positive weight function

$$\tilde{w}(x) = (x-a)(b-x)w(x),$$

then the quadrature formula $\sum_{k=1}^{n-1} W_k f(x_k)$ is exact for all $f = q_{2n-3}$ polynomials of degree less than or equal to $2n-3$, and we can conclude that

$$\int_a^b w(x)p_{2n-1}(x)\,dx = W_0 p_{2n-1}(a) + \sum_{k=1}^{n-1} W_k p_{2n-1}(x_k) + W_n p_{2n-1}(b),$$

is exact for all polynomials of degree less than or equal to $2n-1$.

c) Show that all the weights W_k, $k = 0, 1, \dots, n$, are positive.

Solution. By definition W_0 and W_n are positive as they are obtained integrating positive functions. For $W_1, \dots, W_{n-1}$ we use the well known result saying that Gauss quadrature weights are positive.

Problem 3

a) We want to find the local error σ_{n+1} of the trapezoidal rule method

$$y_{n+1} = y_n + \frac{1}{2}h(f(y_{n+1}) + f(y_n)),$$

for the numerical solution of the scalar initial value problem $y'(t) = f(y)$, with $y(0) = y_0$, and where $h = t_{n+1} - t_n$.

We use the following definition of the local truncation error

$$\sigma_{n+1} = y(t_{n+1}) - z_{n+1},$$

with z_{n+1} defined by

$$z_{n+1} = y(t_n) + \frac{1}{2}h(f(y(t_{n+1})) + f(y(t_n))),$$

and it is sufficient to investigate the case $n = 0$.

Explain how we obtain the following expression for σ_1

$$\sigma_1 := -\frac{1}{2}\int_0^h (h-x)\,x\,y'''(\xi(x))\,dx,$$

and using the mean value theorem for integrals or otherwise find

$$\sigma_1 = -\frac{1}{12}h^3 y'''(\tilde{\xi}),$$

for some $\tilde{\xi}$ in the interval $(0, h)$, where y is the solution of the initial value problem.

Solution. For the exact solution we have

$$y(t_1) = y_0 + \int_0^h f(y(x))\,dx,$$

and z_1 can be interpreted as

$$z_1 = y_0 + \int_0^h g(x)\,dx$$

where $g(x)$ is the linear polynomial interpolating the values $(0, f(y_0))$ and $(h, f(y(t_1)))$. Recall that the error for such interpolation polynomial is

$$f(y(x)) - g(x) = \frac{1}{2!}\left.\frac{d^2 f(y(x))}{dx^2}\right|_{x=\tilde{x}} x\,(x-h) = \frac{1}{2!}y'''(\xi(x))\,x\,(x-h),$$

for $\tilde{x} \in (0, h)$ and where $\xi(x) = \tilde{x}$. This yields the first given expression for σ_1. Since $(h-x)\,x$ is nonnegative, by the mean value theorem for integrals there exists $\tilde{\xi} \in (0, h)$ such that

$$\sigma_1 = -\frac{1}{2}y'''(\tilde{\xi})\int_0^h (h-x)\,x\,dx,$$

and the final result is obtained by computing the integral.

b) Suppose f satisfies the Lipschitz condition

$$|f(t,u) - f(t,v)| \leq L|u - v|,$$

for all real t, u, v where L is a positive constant independent of t, and that $|y'''(t)| \leq M$ for some positive constant M independent of t. Show that the global error $e_n = y(t_n) - y_n$ satisfies the inequality

$$|e_{n+1}| \leq \frac{h^3 M}{12} + (1 + \frac{1}{2}hL)|e_n| + \frac{1}{2}hL|e_{n+1}|.$$

Hint. Use that $e_{n+1} = y(t_{n+1}) - z_{n+1} + z_{n+1} - y_{n+1} = \sigma_{n+1} + z_{n+1} - y_{n+1}$.

Solution. Using the Lipschitz condition we observe that

$$|z_{n+1} - y_{n+1}| \leq |e_n| + \frac{1}{2}hL|e_{n+1}| + \frac{1}{2}hL|e_n|,$$

which substituted in

$$|e_{n+1}| \leq |\sigma_{n+1}| + |z_{n+1} - y_{n+1}|,$$

and together with $|\sigma_{n+1}| \leq \frac{h^3 M}{12}$, gives the desired result.

c) For a constant step-size $h > 0$ satisfying $hL < 2$, deduce that, if $y_0 = y(0)$, then

$$|e_n| \leq \frac{h^2 M}{12L} \left[\left(\frac{1 + \frac{1}{2}hL}{1 - \frac{1}{2}hL} \right)^n - 1 \right].$$

Solution. Since $1 - \frac{1}{2}hL > 0$, from the result of the previous exercise we obtain

$$|e_{n+1}| \leq \frac{1}{1 - \frac{1}{2}hL} \left((1 + \frac{1}{2}hL)|e_n| + \frac{1}{12}h^3 M \right).$$

Let us define $Y := \frac{h^3 M}{12(1-\frac{1}{2}hL)}$ and $X := \frac{1+\frac{1}{2}hL}{1-\frac{1}{2}hL}$, so

$$|e_n| \leq Y + X|e_{n-1}| \leq Y + XY + X^2 Y + \cdots + X^{n-1}Y,$$

because $e_0 = 0$, and using the formula for the partial sums of the geometric series

$$|e_n| \leq Y \frac{X^n - 1}{X - 1}.$$

Since $(X-1)^{-1} = \frac{1-\frac{1}{2}hL}{hL}$ one easily obtains the desired inequality.

Formulae and useful results

- **Partial sums of the geometric series**: for $x \neq 0$,

$$1 + x + x^2 + \cdots + x^m = \frac{1 - x^{m+1}}{1 - x}.$$

- **Mean value theorem for integrals.** Let $f(x)$ and $g(x)$ be continuous on $[a, b]$. Assume that $g(x)$ is positive, i.e. $g(x) \geq 0$ for any $x \in [a, b]$. Then there exists $c \in (a, b)$ such that

$$\int_a^b f(t)g(t)dt = f(c) \int_a^b g(t)dt.$$