Norwegian University of Science and Technology
Department of Mathematical Sciences

Semester Project TMA 4215 - Part 1

Deadline September 9th 2012

Problem 0 This task is voluntary, and nothing should be handed in.

Start Matlab. The first thing you should do is to spend some time familiarizing yourself with the user interface. The main windows you will be using is the command window and a text editor. Matlab has its own built-in editor which we recommend that you use, however if you rather want to use one that you are familiar with, that is also fine. The main reason to use the built-in editor is that it has a debugger, which will certainly come in handy when you try to find bugs in your programs.

Matlab is a powerful tool, with tons of functionality and features. The main things you should be able to handle are:

- Use of scalars, vectors and matrices and how to do basic operations on these.
- Flow-control, mainly `if`-statements and `for`-loops.
- Functions and scripts.
- Simple plots.
- And, most importantly, how to utilize Matlab's help functions `help`, `doc` and `lookfor`.

Matlab's introduction videos are handy for getting started. You can find these by selecting Help→Product Help in the menubar of Matlab's main window. The videos are found under the header “Getting Started”. For this course, the videos “Getting Started” and “Writing a Matlab Program” are most relevant.

Watch the videos “Getting Started” and “Writing a Matlab Program”.

Problem 1 For this problem, you are to use Matlab for a selection of basic numerical tasks.

Your answer to this problem should include

- Printout of Matlab-commands and results for part **a)** and **b)** with relevant comments, for instance like this:

```
>> a=2;
>> b=2;
>> a+b

ans =

     5
>> % Calculation approved by Minitrue.
```

- The scripts you wrote for part **c)** and **d)**.
- The MATLAB-commands you have used to run the program in part **e)**.
- One plot from part **e)**.

a) Given

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 3 \\ 2 & 4.5 & 6 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ -3.5 \end{bmatrix}.$$

Solve the linear system of equations $A\mathbf{x} = \mathbf{b}$ in MATLAB. Check that the answer is correct.

b) Let A be as in **a)**. Find a lower triangular matrix L and an upper triangulær matrix U such that $LU = A$. The matrices L, U are called the *LU-decomposition.* In this task you have to find the correct MATLAB-command yourself.

In the following tasks you are asked to write small MATLAB programs on your own. If you are a programming novice, my advice is: *Think slowly!* Do not write a single line of code before you are certain you know what the program is supposed to do, and how it should do it. If you think doing the task by hand or calculator first would help, that is what you should do.

c) In introductory calculus you learned Newton's method for finding the roots of nonlinear equations, i.e. how to solve $f(x) = 0$ for a nonlinear $f(x)$. The method is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, \ldots$$

Write a script where you solve the fifth degree equation

$$x^5 + x + 1 = 0$$

by using Newton's method. Use $x = 0$ as your initialvalue, and do four iterations.

d) Modify the script from **c)** to use a *stopping criterion.* Specifically, the iterations should continue until the absolute value of the function value is less then a given tolerance, here

$$|f(x_n)| < 10^{-8}.$$

e) `euler.m` on the course web page is a small program which calculates an approximate solution to ordinary differential equations on the form $\mathrm{d}y/\mathrm{d}t = f(t, y)$ by the forward Euler method. Use the program to find an approximate solution for

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} y_1 &= -\sin y_2, & y_1(0) &= 1.5 \\ \frac{\mathrm{d}}{\mathrm{d}t} y_2 &= y_1, & y_2(0) &= 0. \end{aligned}$$

for $0 \le t \le 10$. Use $h = 0{,}1$. Plot your results.

Problem 2

a) The vector function $\mathbf{x} \mapsto \mathbf{f}(\mathbf{x})$ of two variables is defined by

$$f_1(x_1, x_2) = x_1^2 + x_2^2 - 2, \quad f_2(x_1, x_2) = x_1 - x_2.$$

Verify that the equation $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ has two solutions $x_1 = x_2 = 1$ and $x_1 = x_2 = -1$. Show that if $x_1^{(0)} + x_2^{(0)} \neq 0$, then one iteration of Newton's method for the solution of this system gives $\mathbf{x}^{(1)} = (x_1^{(1)}, x_2^{(1)})^T$, with

$$x_1^{(1)} = x_2^{(1)} = \frac{\left(x_1^{(0)}\right)^2 + \left(x_2^{(0)}\right)^2 + 2}{2\left(x_1^{(0)} + x_2^{(0)}\right)}.$$

Prove convergence and deduce that the iteration converges to $(1, 1)^T$ if $x_1^{(0)} + x_2^{(0)}$ is positive, and, if $x_1^{(0)} + x_2^{(0)}$ is negative, the iteration converges to the other solution.

b) Prove that Newton's method applied to the problem in part **a)** has quadratic convergence.

Problem 3

a) Find the rank of the matrix $\mathbf{u}\mathbf{v}^\top$, where $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$.

b) Show that the inverse of the matrix

$$A = I - \mathbf{u}\mathbf{v}^\top$$

where I is the $n \times n$ identity matrix, $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$ og $\mathbf{v}^\top\mathbf{u} \neq 1$, can be written on the form

$$A^{-1} = I + \gamma \mathbf{u}\mathbf{v}^\top.$$

Find γ.

c) What can you say about A if $\mathbf{v}^\top\mathbf{u} = 1$?

d) Find an upper bound for the condition number

$$\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2,$$

in terms of norms of $\mathbf{u}$, $\mathbf{v}$ and the inner product $\mathbf{v}^\top \mathbf{u}$.

e) What is the inverse of the matrix

$$I - \alpha E^{k,l}$$

where $E^{k,l}$ is the matrix with 1 as entry k, l and 0 otherwise.

f) Matrices of the form $I - \alpha E^{k,l}$ corresponds to elementary row operations, such that if $B' = (I - \alpha E^{k,l})B$, then B' is the matrixobtained by subtracting a multiplum of row l from row k in B. Let $B^{(0)}$ be an $n \times n$-matrix without zeros on the diagonal, and let $1 \leq q < p \leq n$. Determine α, k and l suck that

$$B^{(1)} = A^{(0)} B^{(0)} = (I - \alpha E^{k,l}) B^{(0)}$$

has a 0 as entry p, q. Show that there is a sequence of matrices, $A^{(0)}, A^{(1)}, \ldots A^{(r)}$ all of which on the form, $I - \alpha E^{k,l}$ such that column p in the matrix

$$B^{(r+1)} = A^{(r)} \cdots A^{(0)} B^{(0)}$$

has only zeros below row p.

g) Determine

$$A = A^{(r)} \cdots A^{(0)}$$

where $A^{(0)}, \ldots, A^{(r)}$ are as in part **f)**. Also determine A^{-1}.

h) Does there exist a matrix $C^{k,l}$ corresponding to interchanging row k and l? That is, matrix $C^{k,l}$ is such that if $B' = C^{k,l} B$ then B' is the matrix obtained by interchanging row k and l in B.

i) We now let $\mathbf{u} = \mathbf{v}$ and consider the symmetric matrix.

$$A = I - \mathbf{u}\mathbf{u}^\top$$

What are the eigenvalues and eigenvectors of A? Determine the condition number $\kappa_2(A)$ and compare with the upper bound from **d)**.

j) For some vectors $\mathbf{u}$, $A = I - \mathbf{u}\mathbf{u}^\top$ is orthogonal. (A matrix B is orthogonal if $B^\top B = I$). What are necessary and sufficient conditions on $\mathbf{u}$ for A to be orthogonal?