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Exam in TMA4215

August 17th 2011, Time: 9:00-13:00

Hjelpemidler code C: Textbook Kincaid and Cheney, Numerical Analysis, third edition. TMA4215 lecture notes (39 pages).

Problem 1

a) Is the following function a natural cubic spline?

$$S(x) = \begin{cases} x^3 - 1 & x \in [-1, \frac{1}{2}] \\ 3x^3 - 1 & x \in [\frac{1}{2}, 1] \end{cases}$$

Justify your answer.
Answer No. First of all it does not satisfy the condition for being natural (the second derivative should be equal to 0 in -1 and 1). Moreover it is not continuous in $1/2$ which implies it is not a spline.

b) What values of (a, b, c, d) make the following a cubic spline?

$$f(x) = \begin{cases} x^3 & x \in [-1, 0] \\ a + bx + cx^2 + dx^3 & x \in [0, 1] \end{cases}$$

Answer. For all values $(0, 0, 0, d)$, where d is an arbitrary real value.

Problem 2

a) Find the first two polynomials orthogonal with respect to the inner product

$$\langle f, g \rangle_w = \int_0^1 w(x)\, f(x) g(x)\, dx,$$

where $w(x) = x$ in $[0, 1]$ is the weight function. Verify that the third orthogonal polynomial is

$$P_2(x) = \sqrt{3}\,(10x^2 - 12x + 3).$$

Note that we have chosen the normalization $\langle P_2, P_2 \rangle_w = 1$, use the same normalization for the other two polynomials P_0 and P_1.
Answer. $P_0 \equiv \sqrt{2}$, $P_1 = 4 - 6x$.

b) Find a quadrature formula of the form

$$\int_0^1 xf(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

with $n = 1$, that is exact for all polynomials of degree 3.
Answer. For $n = 1$ the Gaussian quadrature formulae are exact for all polynomials of degree less than or equal to $2n + 1 = 3$. The answer to this question is then the Gauss quadrature formula for the weight function $w(x) \equiv x$ on $[0, 1]$, we will construct this quadrature formula. By theorem on Gaussian quadrature (page .. in Kincaid and Cheney), the nodes of the quadrature are the zeros of P_3, that is $x_0 = \frac{6-\sqrt{6}}{10}$ and $x_1 = \frac{6+\sqrt{6}}{10}$. We then use the formula for the weights (page .. in Kincaid and Cheney), and we get

$$A_0 = \frac{1}{4} - \frac{1}{36}\sqrt{6}, \quad A_1 = \frac{1}{4} + \frac{1}{36}\sqrt{6}.$$

c) Use this formula to approximate the integral

$$\int_0^1 x\sin(x)\,dx.$$

Compute the error subtracting the resulting approximation from the exact integral.
Answer. We apply the formula to approximate the given integral and we get

$$\int_0^1 x\sin(x)\,dx \approx \big(\frac{1}{4} - \frac{1}{36}\sqrt{6}\big)\sin\big(\frac{6-\sqrt{6}}{10}\big) + \big(\frac{1}{4} + \frac{1}{36}\sqrt{6}\big)\sin\big(\frac{6+\sqrt{6}}{10}\big).$$

Calculating the integral at the left hand side and the obtained value of the right hand side we get

$$\sin(1) - \cos(1) = 0.3011686 \approx 0.3011307.$$

Giving an error

$$\int_0^1 x\sin(x)\,dx - A_0\sin(x_0) - A_1\sin(x_1) = -3.79 \cdot 10^{-5}.$$

Problem 3

a) We want to find the local error σ_{n+1} of the method

$$y_{n+1} = y_n + \frac{1}{2}h\big(f(y_n) + f(y_n + hf(y_n))\big),$$

for the numerical solution of the autonomous, scalar initial value problem $y'(t) = f(y(t))$, with $y(0) = y_0$, and where $h = t_{n+1} - t_n$.

We use the following definition of the local truncation error

$$\sigma_{n+1} = y(t_{n+1}) - z_{n+1},$$

with z_{n+1} defined by

$$z_{n+1} = y(t_n) + \frac{1}{2}h\big(f(y(t_n)) + f(y(t_n) + hf(y(t_n)))\big),$$

and it is sufficient to investigate the case $n = 0$.
Explain how we obtain the following expression for σ_1

$$\sigma_1 = h^3(C_1 f''(y_0)[f(y_0)]^2 + C_2[f'(y_0)]^2 f(y_0)) + \mathcal{O}(h^4)$$

find the constants C_1 and C_2 [1].
Answer. By Taylor expansion we get for the exact solution

$$y(h) = y_0 + hf(y_0) + \frac{h^2}{2}f'(y_0)f(y_0) + \frac{h^3}{3!}\left(f''(y_0)f(y_0)^2 + [f'(y_0)]^2 f(y_0)\right) + \mathcal{O}(h^4)$$

and analogously for the numerical solution

$$z_1 = y_1 = y_0 + \frac{1}{2}h\left(f(y_0) + f(y_0) + hf'(y_0)f(y_0) + \frac{h^2}{2}f''(y_0)f(y_0)^2 + \mathcal{O}(h^3)\right)$$

and then

$$y(h) - z_1 = h^3\left(\frac{1}{6}f''(y_0)f(y_0)^2 + \frac{1}{6}[f'(y_0)]^2 f(y_0) - \frac{1}{4}f''(y_0)f(y_0)^2\right) + \mathcal{O}(h^4)$$

$$y(h) - z_1 = h^3\left(-\frac{1}{3}f''(y_0)f(y_0)^2 + \frac{1}{6}[f'(y_0)]^2 f(y_0)\right) + \mathcal{O}(h^4),$$

so $C_1 = -\frac{1}{12}$ and $C_2 = \frac{1}{6}$.

b) We will use the method for the numerical solution of the initial value problem

$$y'' = y'\,y, \quad y(0) = 1, \quad y'(0) = 0.5.$$

Reformulate the problem into a system of first order differential equations. Do one step with the proposed method to find the numerical approximations to $y(0.1)$ and $y'(0.1)$.
Answer. The given differential equation gives rise to the first order system

$$\begin{cases} y' &= v, \quad y(0) = 1, \\ v' &= vy, \quad v(0) = 0.5. \end{cases}$$

We define

$$\mathbf{y} := \begin{bmatrix} y \\ v \end{bmatrix},$$

[1] If one considers $f : \mathbf{R} \to \mathbf{R}$, then $f'(y) = \frac{d}{dy}f(y)$ and $f''(y) = \frac{d^2}{dy^2}f(y)$. Otherwise for $f : \mathbf{R}^k \to \mathbf{R}^k$, $f'(y)$ is the Jacobian of f and f'' is the second differential.

and use the method, we get

$$\mathbf{y}_1 = \mathbf{y}_0 + \frac{h}{2}\left(f(\mathbf{y}_0) + f(\mathbf{y}_0 + hf(\mathbf{y}_0))\right)$$

and here f represents the right hand side of the system. We have $\mathbf{y}_0 = [1\,,\,0.5]^T$ and $h = 0.1$, and substituting into the formula we get

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} + \frac{0.1}{2}\left(\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} + \begin{bmatrix} 0.55 \\ 0.55 \cdot 1.05 \end{bmatrix}\right) = \begin{bmatrix} 1.0525 \\ 0.5539 \end{bmatrix}.$$