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Exam in TMA4215

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Hjelpemidler code C: Textbook Kincaid and Cheney, Numerical Analysis, third edition. TMA4215 lecture notes (39 pages).

Problem 1

a) Find the polynomial p of degree less than or equal to 3, such that

$$p(0) = 1, \quad p(1) = -1, \quad p'(0) = 1, \quad p''(0) = 0.$$

Problem 2

a) Let $n \geq 2$. Show that any polynomial p_{2n-1} of degree $2n-1$ can be written

$$p_{2n-1}(x) = (x-a)(b-x)q_{2n-3}(x) + r(x-a) + s(b-x),$$

where q_{2n-3} is a polynomial of degree $2n-3$, and a, b, r and s are constants.

Hint. Observe that the set of polynomials

$$(x-a), \quad (b-x), \quad (x-a)\cdot(b-x)\cdot x^k, \quad k = 0,\dots,2n-3,$$

with a and b not simultaneously equal to zero, is a basis for the vector space of polynomials of degree less than or equal to $2n-1$.

b) Then construct the Lobatto quadrature formula

$$\int_a^b w(x)f(x)\,dx \approx W_0 f(a) + \sum_{k=1}^{n-1} W_k f(x_k) + W_n f(b),$$

which is exact when f is a polynomial of degree $2n-1$. Give a matematical description of the nodes and the weights. Here $w(x)$ is a positive weight function.

Hint. One way to solve this problem is by using the $n-1$ Gauss quadrature points and weights with respect to a weight function $\tilde{w}(x) \neq w(x)$.

c) Show that all the weights W_k, $k = 0, 1, \dots, n$, are positive.

Problem 3

a) We want to find the local error σ_{n+1} of the trapezoidal rule method

$$y_{n+1} = y_n + \frac{1}{2}h(f(y_{n+1}) + f(y_n)),$$

for the numerical solution of the scalar initial value problem $y'(t) = f(y)$, with $y(0) = y_0$, and where $h = t_{n+1} - t_n$.

We use the following definition of the local truncation error

$$\sigma_{n+1} = y(t_{n+1}) - z_{n+1},$$

with z_{n+1} defined by

$$z_{n+1} = y(t_n) + \frac{1}{2}h(f(y(t_{n+1})) + f(y(t_n))),$$

and it is sufficient to investigate the case $n = 0$.

Explain how we obtain the following expression for σ_1

$$\sigma_1 := -\frac{1}{2}\int_0^h (h - x)\, x\, y'''(\xi(x))\, dx,$$

and using the mean value theorem for integrals or otherwise find

$$\sigma_1 = -\frac{1}{12}h^3 y'''(\tilde{\xi}),$$

for some $\tilde{\xi}$ in the interval $(0, h)$, where y is the solution of the initial value problem.

b) Suppose f satisfies the Lipschitz condition

$$|f(t, u) - f(t, v)| \leq L|u - v|,$$

for all real t, u, v where L is a positive constant independent of t, and that $|y'''(t)| \leq M$ for some positive constant M independent of t. Show that the global error $e_n = y(t_n) - y_n$ satisfies the inequality

$$|e_{n+1}| \leq \frac{h^3 M}{12} + (1 + \frac{1}{2}hL)|e_n| + \frac{1}{2}hL|e_{n+1}|.$$

Hint. Use that $e_{n+1} = y(t_{n+1}) - z_{n+1} + z_{n+1} - y_{n+1} = \sigma_{n+1} + z_{n+1} - y_{n+1}$.

c) For a constant step-size $h > 0$ satisfying $hL < 2$, deduce that, if $y_0 = y(0)$, then

$$|e_n| \leq \frac{h^2 M}{12L}\left[\left(\frac{1 + \frac{1}{2}hL}{1 - \frac{1}{2}hL}\right)^n - 1\right].$$

Formulae and useful results

- **Partial sums of the geometric series**: for $x \neq 0$,

$$1 + x + x^2 + \cdots + x^m = \frac{1 - x^{m+1}}{1 - x}.$$

- **Mean value theorem for integrals.** Let $f(x)$ and $g(x)$ be continuous on $[a, b]$. Assume that $g(x)$ is non-negative, i.e. $g(x) \geq 0$ for any $x \in [a, b]$. Then there exists $c \in (a, b)$ such that

$$\int_a^b f(t)g(t)dt = f(c)\int_a^b g(t)dt.$$