Norwegian University of Science and Technology
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TMA4212 Numerical Solution of Differential Equations by Difference Methods
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Exercise set 4 Solutions

1 A boundary value problem for the transport equation

a) We rewrite the scheme as :

$$\text{(S)} \qquad \underbrace{1}_{\alpha_P} U_m^{n+1} - \underbrace{(1-r)}_{\alpha_S} U_m^n - \underbrace{r}_{\alpha_{SW}} U_{m-1}^n = 0, \qquad r = 3\frac{k}{h},$$

We have monotonicity if:

- $\alpha_P \geq 0$, which is always true;
- $\alpha_S \geq 0$, which is true if $1 - r > 0 \rightarrow r < 1 \rightarrow \frac{k}{h} < \frac{1}{3}$, which represents our CFL condition;
- $\alpha_p = 1 = \sum_{Q, Q \neq P} \alpha_Q$, which is always true.

The same can be verified on the boundary.

b) I choose $\phi = t\|f\|_{L^\infty((0,1)\times(0,T))}$ as comparison function. Following standard arguments seen in class I have that:

$$(1) \qquad -L_h V_m^n - \left(-L_h \left(t\|f\|_{L^\infty((0,1)\times(0,T))}\right)\right) = f_m^{n+1} - \|f\|_{L^\infty((0,1)\times(0,T))} \leq 0,$$

The application of the maximum principle allows me to conclude that the maximum of $V_m^n - t^n\|f\|_{L^\infty((0,1)\times(0,T))}$ (and, with analogous procedure $-V_m^n - t^n\|f\|_{L^\infty((0,1)\times(0,T))}$) is either on the boundary or 0. Since $V_m^n|_{\partial_{IF}\Omega} = 0$ on the boundaries and $-t_n\|f\|_{L^\infty((0,1)\times(0,T))}$ is negative, we have that

$$\max\left[\pm V_m^n - t^n\|f\|_{L^\infty((0,1)\times(0,T))}\right] \leq 0$$

. Then:

$$\max_{\Omega}\left[\pm V_m^n - t^n\|f\|_{L^\infty((0,1)\times(0,T))}\right] \leq 0 \rightarrow \pm V_m^n \leq t_n\|f\|_{L^\infty((0,1)\times(0,T))} \leq T\|f\|_{L^\infty((0,1)\times(0,T))}.$$

and stability with respect to the right hand side is proved. As usual this same procecure can be applied to the error equation $e_m^n = u_m^n - U_m^n$ to prove $e_m^n \leq T\|\tau\|_{L^\infty((0,1)\times(0,T))}$ where τ_m^n is the truncation error.

c) The scheme uses just first order schemes, so we have that:

$$\begin{aligned}
\tau_m^n &= \frac{u_m^{n+1} - u_m^n}{k} + 3\frac{u_m^n - u_{m-1}^n}{h} - (u_t + 3u_x) \\
&= u_t + \frac{k}{2}u_{tt} + \mathcal{O}(k^2) + 3u_x + \frac{3h}{2}u_{xx} + \mathcal{O}(h^2) - (u_t + 3u_x) \\
&= \frac{k}{2}u_{tt}(\xi_1) + \frac{3h}{2}u_{xx}(\xi_2) \leq \frac{k}{2}\|u_{tt}\|_{L^\infty((0,1)\times(0,T))} + \frac{3h}{2}\|u_{xx}\|_{L^\infty((0,1)\times(0,T))}.
\end{aligned}$$

for some ξ_i. We thus have:

$$C_1 = \frac{1}{2}\|u_{tt}\|_{L^\infty((0,1)\times(0,T))}, \quad C_2 = \frac{3}{2}\|u_{xx}\|_{L^\infty((0,1)\times(0,T))}.$$

d) We define a grid with stepsize $h = 1/M$ in the x direction and $k = 1/N$ in the y direction. We have to solve for points $P_{m,n} = (mh, nk)$ with $m = 1, \dots, M$ and $n = 1, \dots, N$. With reference to (S) we will have just one boundary case (excluding the initial step):

1. U^n_{m-1} is known, the scheme becomes:

$$U^{n+1}_m - (1-r)U^n_m = kf^n_m + rf(P_{m-1,n}).$$

2 A variable coefficient transport equation

a) The Upwinding procedure is decided by the sign of $a(x_m, t_n)$. If $a(x_m, t_n) > 0$ then the scheme is the same as in (S), but if $a(x_m, t_n) < 0$ then I have to take the spatial difference in the other direction and I obtain:

$$\text{if } a(x_m, t_n) > 0: \quad U^{n+1}_m - (1 - r^n_m)U^n_m - r^n_m U^n_{m-1} = 0,$$
(S2)
$$\text{if } a(x_m, t_n) < 0: \quad U^{n+1}_m - (1 + r^n_m)U^n_m + r^n_m U^n_{m+1} = 0, \qquad r^n_m = a(x_m, t_n)\frac{k}{h}.$$

b) To check Von Neumann stability we rewrite the scheme in a unified way as:

$$U^{n+1}_m - (1 - r^+ - r^-)U^n_m + r^+U^n_{m-1} + r^-U^n_{m+1} = 0$$

where:

$$r^+ = \max(r, 0), \quad r^- = (-r)^+, \quad r = r^+ - r^-.$$

We now identify $U^n_m = \xi^n e^{i\beta x_m}$ and the scheme above reads:

$$\xi^{n+1}e^{i\beta x_m} - (1 - r^+ - r^-)\xi^n e^{i\beta x_m} + r^+\xi^n e^{i\beta(x_m - h)} + r^-\xi^n e^{i\beta(x_m+h)} = 0$$

which simplifies to:

$$\begin{aligned} \xi &= (1 - r^+ - r^-) - r^+e^{-i\beta h} - r^-e^{i\beta h} \\ &= (1 - |r|) - |r|\cos(\beta h) - ir\sin(\beta h) \end{aligned}$$

Taking the square of the modulus leads to:

$$\begin{aligned} |\xi|^2 &= 1 + r^2 + r^2\cos(\beta h)^2 - 2|r| - 2|r|\cos(\beta h) + 2r^2\cos(\beta h) + r^2\sin(\beta h)^2 \\ &= 1 + 2r^2 - 2|r| + [2r^2 - 2|r|]\cos(\beta h) = 1 + (2r^2 - 2|r|)(1 + \cos(\beta h)) \end{aligned}$$

Von Neumann stability is given if:

$$|\xi|^2 < 1 \to (2r^2 - 2|r|)(1 + \cos(\beta h)) < 0 \to 2r^2 - 2|r| < 0 \to -1 < r < 1$$

or, alternatively

$$k < \frac{h}{|a|}.$$

To check if the scheme is dissipative or dispersive we similarly indetify $U_m^n = \rho^n e^{i(\omega t_n + \beta x_m)}$ obtaining:

$$\rho^{n+1} e^{i(\omega t_n + \beta x_m)} - (1 - r^+ - r^-)\rho^n e^{i(\omega t_n + \beta x_m)} \\ + r^+ \rho^n e^{i(\omega t_n + \beta(x_m - h))} + r^- \rho^n e^{i(\omega t_n + \beta(x_m + h))}$$

and simplifying:

$$\begin{aligned}\rho &= (1 - r^+ - r^-) - r^+ e^{-i\beta h} - r^- e^{i\beta h} \\ &= (1 - |r|) - |r|\cos(\beta h) - ir\sin(\beta h) = 0\end{aligned}$$

then

$$|\rho|^2 = 1 + (2r^2 - 2|r|)(1 + \cos(\beta h))$$

and since $\rho < 1$ by the stability assumption made the scheme is dissipative. Also, we have:

$$\arg(\rho) = \arctan\left[-\frac{r\sin(\beta h)}{(1 - |r|) - |r|\cos(\beta h)}\right]$$

which is nonlinear in βh and thus the scheme is also dispersive.

c) The Leap-frog scheme for the problem at hand reads:

$$\frac{U_m^{n+1} - U_m^{n-1}}{2k} + a(x_m, t_n)\frac{U_{m+1}^n - U_{m-1}^n}{2h} = 0.$$

Rearranging:

$$U_m^{n+1} - U_m^{n-1} + rU_{m+1}^n - rU_{m-1}^n = 0.$$

We have now two types of boundary conditions. On $x = 0$:

$$U_1^{n+1} - U_1^{n-1} + rU_2^n = rf(P_{0,n}).$$

On $x = 1$, as suggested, we extend the value found for U_{M-1}^n at the time step before outside the domain and impose:

$$U_{M-1}^{n+1} - U_{M-1}^{n-1} + rU_{M-1}^n - rU_{M-2}^n = 0.$$