Norwegian University of Science and Technology
Department of Mathematical Sciences

Semester Project 2

1 Biharmonic equation (Abdulhaque)

Consider the inhomogeneous Biharmonic equation with clamped boundary conditions on the unit square $\Omega = [0,1]^2$:

$$\nabla^4 u = f \quad , \quad (x,y) \in \Omega \tag{1a}$$
$$u = 0 \ , \ \nabla^2 u = 0 \quad , \quad (x,y) \in \partial\Omega \tag{1b}$$

In this setting, we have two partial differential operators:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad , \qquad \nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

a) Justify that the solution of this BVP can be represented by a double Fourier sine series on the form

$$u(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{mn} \sin(m\pi x)\sin(n\pi y) \tag{2}$$

Calculate the coefficient F_{mn}. How can you choose f such that the infinite series collapse into a single expression? (Hint: Euler's orthogonality relations.)

b) Transform (1) into a system of Poisson equations.

c) Consider the 5-point stencil ($\nabla_5^2 u$) and 9-point stencil ($\nabla_9^2 u$) for $\nabla^2 u = f$:

$$u_{i-1,j} + u_{i+1,j} - 4u_{ij} + u_{i,j-1} + u_{i,j+1} = h^2 f_{ij} \tag{3}$$

$$\begin{aligned} &\frac{1}{6}\left(u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i-1,j-1}\right) \\ &+ \frac{3}{2}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right) - \frac{10}{3}u_{ij} \\ &= h^2\left[\frac{2}{3}f_{ij} + \frac{1}{12}\left(f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1}\right)\right] \end{aligned} \tag{4}$$

Describe in full detail what the system matrix, consisting of several blocks, will look like. What are the eigenvalues of the TST-matrices in this system? (Hint: TST is an abbreviation for Toeplitz, Symmetric and Tridiagonal. They have many special properties you can take advantage of in the later numerical computation process.)

d) Prove that the 5-point stencil is stable and of order 2. Follow this procedure:

1. Discrete maximum principle: If $\nabla_5^2 v \geq 0$ on Ω, then the maximum value of v on $\overline{\Omega}$ attains its value on $\partial\Omega$.
2. If v_h is a discrete function that equals 0 on $\partial\Omega$, then

$$\|v\|_\infty \leq \frac{1}{8}\|\nabla_5^2 v\|_\infty$$

3. If $\nabla^2 u = f$ and $\nabla_5^2 v = f$, then

$$\|u - v\|_\infty \leq Ch^2 |D^4 v|_\infty$$

e) Prove that the 9-point stencil is stable and of order 4. Follow this procedure:

1. Discrete maximum principle: If $\nabla_9^2 v \geq 0$ on Ω, then the maximum value of v on $\overline{\Omega}$ attains its value on $\partial\Omega$.
2. If v_h is a discrete function that equals 0 on $\partial\Omega$, then

$$\|v\|_\infty \leq \frac{3}{40}\|\nabla_9^2 v\|_\infty$$

3. If $\nabla^2 u = f$ and $\nabla_9^2 v = f$, then

$$\|u - v\|_\infty \leq Ch^4 |D^6 v|_\infty$$

f) Describe how the Fast Poisson Solver method can be used, and give a concise explanation of how it works. (Hint: Fast Fourier Sine Transform.)

g) Perform uniform mesh refinement with $M = \{8, 16, 32, 64, 128, 256\}$ in order to verify the order of the 5-point and 9-point stencils. Apply 1a) to choose an analytical solution, and use the same number of grid points in both directions.

h) Consider the following BVP:

$$\begin{aligned} \nabla^4 u &= f(x, y) & \qquad (x, y) &\in \Omega \\ u = 0\,,\ \nabla^2 &= 0 & \qquad (x, y) &\in \partial\Omega \end{aligned}$$

where $f(x, y) = (\sin(\pi x)\sin(\pi y))^4 e^{-(x-0.5)^2-(y-0.5)^2}$. Solve the given problem as good as you can and report the results in the following loglog-graphs:

- x-axis with degrees of freedom, y-axis for relative l^2-error.
- x-axis with degrees of freedom, y-axis for computational time.

2 Sine-Gordon equation (Abdulhaque)

Consider the Sine-Gordon equation on $\Omega \times I$, where $\Omega = [a,b]$ and $I = [0,T]$:

$$u_{tt} - u_{xx} + \sin(u) = 0 \quad , \quad (x,t) \in \Omega \times I \tag{5a}$$

$$u(a,t) = f_1(t)\ ,\ u(b,t) = f_2(t) \quad , \quad (x,t) \in \partial\Omega \times I \tag{5b}$$

$$u(x,0) = u_0(x)\ ,\ u_t(x,0) = u_1(x) \quad , \quad x \in \Omega \times \{t = 0\} \tag{5c}$$

a) Find the analytical solution with the following procedure:

1. Introduce $s = x - ct$ and express the equation in terms of s.
2. Use the chain rule to integrate.
3. Choose 1 and 0 as the constants of integration.
4. Apply the following identities:

$$\frac{1+\cos(x)}{2} = \sin^2\left(\frac{x}{2}\right) \quad , \quad \int \frac{1}{\sin(x)}\,dx = \ln\tan\left(\frac{x}{2}\right) + C$$

b) Show that the energy is conserved when $u_x(a,t), u_x(b,t) = 0$, i.e. $E(t) = E(0)$. (Hint: Multiply the equation with u_t and integrate with respect to x and t)

c) Use semi-discretization (2nd order stencil on u_{xx}) on the equation. Show that we have two systems of equations (1st and 2nd order in time).

d) Implement the following explicit Runge-Kutta schemes:

0	0	0
1	1	0
	$\frac{1}{2}$	$\frac{1}{2}$

0	0	0	0
$\frac{1}{2}$	$\frac{1}{2}$	0	0
1	-1	2	0
	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$

0	0	0	0	0
$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	0	0
1	0	0	1	0
	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

Table 1: RK2 (order 2), RK3 (order 3), RK4 (order 4).

Perform h- and t-refinement for each integrator (use analytical solution).

e) Implement the following explicit and symplectic Runge-Kutta-Nyström schemes:

0	0
$\bar{b}_i$	$\frac{1}{2}$
b_i	1

$\frac{1}{2} - \delta$	0	0	0
$\frac{1}{2}$	$\frac{1}{24\delta}$	0	0
$\frac{1}{2} + \delta$	$\frac{1}{12\delta}$	$\delta - \frac{1}{12\delta}$	0
$\bar{b}_i$	$b_1(1-c_1)$	$b_2(1-c_2)$	$b_3(1-c_3)$
b_i	$\frac{1}{24\delta^2}$	$1 - \frac{1}{12\delta^2}$	$\frac{1}{24\delta^2}$

Table 2: RKN-12 (order 2), RKN-34 (order 4).

where the generating coefficient is $\delta = \frac{1}{12}(2 - \sqrt[3]{4} - \sqrt[3]{16})$.
Perform h- and t-refinement for each integrator (use analytical solution).

f) Consider the following BVP, where $\Omega = [-2, 2]$ and $I = [0, 4]$:

$$u_{tt} - u_{xx} + \sin(u) = 0 \quad , \quad (x,t) \in \Omega \times I \tag{6a}$$
$$u(-2,t) = 0 \ , \ u(2,t) = 0 \quad , \quad (x,t) \in \partial\Omega \times I \tag{6b}$$
$$u(x,0) = \sin(\pi x)^2 e^{-x^2} \quad , \quad x \in \Omega \times \{t = 0\} \tag{6c}$$
$$u_t(x,0) = \sin(\pi x)^4 e^{-x^2} \quad , \quad x \in \Omega \times \{t = 0\} \tag{6d}$$

Apply the RK4 and RKN-34 integrators, solve the problem as best as you can and report the results in the following two loglog-graphs:

- x-axis with degrees of freedom, y-axis for normalized energy difference, $\Delta E = |E(4) - E(0)|/E(0)$.
- x-axis with degrees of freedom, y-axis for computational time.

Which integrator yields the best result, and what is the reason for this?

Appendix

Sparse matrices (all tasks)

Defining matrices in sparse format is useful for reducing memory storage and computational running time when dealing with linear systems of equations. There are two essential PYTHON-modules for this task:

```
import scipy.sparse as sp
import scipy.sparse.linalg as sl
```

Matrices can be converted between full and sparse format:

```
A = sp.csc_matrix(A)     # Sparse format.
A = A.todense()          # Full format.
```

Banded matrices can be defined as follows:

```
A = sp.diags([-1,2,-1],[1,0,-1],shape=[20,20],format='csc')
B = sp.diags([1,-4,10,-4,1],[2,1,0,-1,-2],shape=[20,20],format='csc')
```

Some typical operations with sparse matrices:

```
c = A*b                  # Matrix-vector multiplication, c = A*b
c = sl.spsolve(A,b)      # Linear equation solving, c = A\b
```

Fast Fourier Transform (Task 1)

The Fast Fourier Transform and its variants are included in the PYTHON-module `scipy`:

```
from scipy.fft import fft, ifft, dst, idst
y = fft(x)               # Fast Fourier Transform
x = ifft(y)              # Inverse Fast Fourier Transform
y = dst(x,type=2)        # Fast Fourier Sine Transform
x = idst(y,type=2)       # Inverse Fourier Sine Transform
```

RK- and RKN-integrators (Task 2)

Runge-Kutta (RK) and Runge-Kutta-Nyström (RKN) integrators are defined as

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b} \end{array} \qquad \begin{array}{c|c} \mathbf{c} & \overline{\mathbf{A}} \\ \hline & \overline{\mathbf{b}} \\ \hline & \mathbf{b} \end{array}$$

Applying an RK-method to $y' = f(y)$ yields

$$\begin{aligned} k_i &= f\left(t_n + c_i h, y_n + h\sum_{j=1}^{s} a_{ij}k_j\right) \quad , \quad 1 \le i \le s \qquad (7) \\ y_{n+1} &= y_n + h\sum_{i=1}^{s} b_i k_i \end{aligned}$$

Applying an RKN-method to $y'' = f(y)$ yields

$$\begin{aligned} k_i &= f\left(t_n + c_i h, y_n + c_i h y_n' + h^2\sum_{j=1}^{s} \overline{a}_{ij}k_j\right) \quad , \quad 1 \le i \le s \qquad (8) \\ y_{n+1}' &= y_n' + h\sum_{i=1}^{s} b_i k_i \\ y_{n+1} &= y_n + h y_n' + h^2\sum_{i=1}^{s} \overline{b}_i k_i \end{aligned}$$