Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4212 Numerical solution of differential equations by difference methods Spring 2021

Exercise set 2

[1] Consider the following heat equation on $x \in [0,1]$,

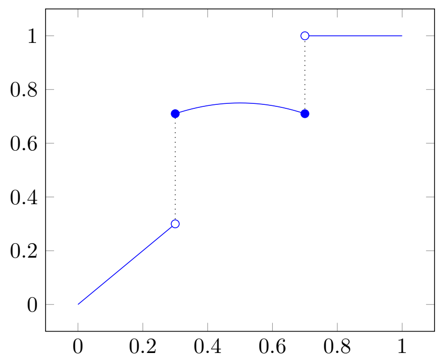
$$u_t = u_{xx}, \quad u(0,t) = u(1,t) = 0, \quad u(0,x) = \begin{cases} 2x, & 0 \le x \le 1/2, \\ 2-2x, & 1/2 \le x \le 1, \end{cases}$$

for $t > 0$ as we considered in Section 4.2.

a) Implement the Crank-Nicolsons method on your computer. Since this is a voluntary exercise, use any language you like. If you choose to use Matlab, the code is given in Section 4.2.4. Observe time evolution of the solution. When time t is sufficiently large, how the solution look like?

[Solution] When time is sufficiently large, the solution converges to zero. Look at the first animation in the supplementary file.

b) The heat equation is known to have smoothing property: even if the initial condition is discontinuous (but in L_2), the solution at any $t > 0$ is in C^∞. Check this property numerically by choosing your own discontinuous initial condition.



[Solution] Look at the second animation in the supplementary file.

c) Modify your code for a general θ method, and compare the numerical solution (your choice of $\theta \neq 1/2$) with the Crank-Nicolsons method. Draw the convergence plot (in terms of M) of this method using as if the Crank-Nicolsons method is the analytic solution, where both methods should use the same number of points M. Theoretically, what convergence rate do you expect? And do you numerically see it?

[Solution] For M large enough, the error difference between first order 1 method ($\theta \neq 1/2$) and order 2 method (Crank-Nicolson) is $\mathcal{O}(M^{-1}) - \mathcal{O}(M^{-2}) = \mathcal{O}(M^{-1})$, therefore we expect the first order convergence and you should actually see it numerically.

d) Consider a modified problem

$$u_t = -u_{xx},$$

with the same boundary/initial conditions. This is known to be an ill-posed problem where the solution diverges analytically. Consider the θ method to numerically solve this problem. Prove that you cannot obtain $F-$stability with any choice of parameters (Hint: look at Section 5.5).

[Solution] Write the linear system in the following manner (look at Section 5.5):

$$(I + \theta r S)\boldsymbol{U}^{n+1} = ((I + (\theta - 1)rS)\boldsymbol{U}^n + \boldsymbol{d}^n,$$

where S is the same as Section 5.5, and our boundary condition is expressed in $\boldsymbol{d}^n$. Note that because of the modified problem, the sign before the matrix S is now different. We want to prove that

$$\rho\left((I + \theta r S)^{-1}(I - (1-\theta)rS)\right) > 1.$$

Now let

$$A := (I + \theta r S) = \text{tridiag}(r\theta, 1 - 2r\theta, r\theta),$$
$$B = (I - (1-\theta)rS) = \text{tridiag}(-r(1-\theta), 1 + 2r(1-\theta), -r(1-\theta)).$$

We know that (e.g., look at the previous exercise) for a tridiagonal matrix tridiag(c, a, b) the eigenvectors $\boldsymbol{x}^{(k)}$ and the associated eigenvalues λ_k are given by

$$x_j^{(k)} = \left(\frac{b}{c}\right)^{j/2} \sin\left(\frac{jk\pi}{M+1}\right), \quad \lambda_k = a + 2\sqrt{bc}\cos\left(\frac{k\pi}{M+1}\right),$$

where $x_j^{(k)}$ is the jth element of the vector $\boldsymbol{x}^{(k)}$; $A\boldsymbol{x}^{(k)} = \lambda_k \boldsymbol{x}^{(k)}$. Observe that A and B have the same eigenvectors. This means we can diagonalize A and B by the same orthogonal matrix T:

$$A = T\Lambda_A T^{-1}, B = T\Lambda_B T^{-1},$$

where Λ_A and Λ_B are diagonal matrices consisting of eigenvalues of A and B, respectively. Therefore,

$$(I + \theta r S)^{-1}(I - (1-\theta)rS) = A^{-1}B = T\Lambda_A^{-1}\Lambda_B T^{-1}.$$

So the eigenvalues of $A^{-1}B$ is

$$\frac{\lambda_{B,k}}{\lambda_{A,k}} = \frac{1 + 2(1-\theta)r + 2(1-\theta)r\cos(k\pi/(M+1))}{1 - 2\theta r + 2\theta r \cos(k\pi/(M+1))},$$

for $k = 1, ..., M$. With easy calculation one can show

$$\forall k \ \frac{\lambda_{B,k}}{\lambda_{A,k}} \geq 1, \text{ and, } \exists k \ \frac{\lambda_{B,k}}{\lambda_{A,k}} > 1.$$

Therefore,

$$\rho(A^{-1}B) > 1.$$

2 Solve the problems 3, 4 and 5 of the exercise 1 from 2020.

[Solution] Solution is given here.