Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4212 Numerical solution of differential equations by difference methods Spring 2021

Exercise set 2

[Updated on 19th Feb]

$\boxed{1}$ Consider the following heat equation on $x \in [0,1]$,

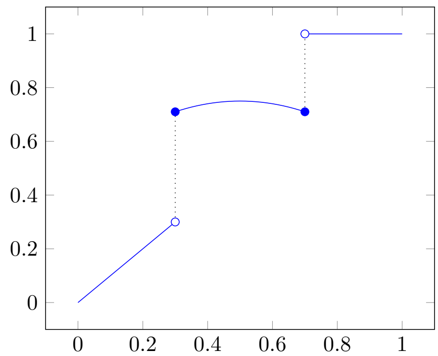
$$u_t = u_{xx}, \quad u(0,t) = u(1,t) = 0, \quad u(0,x) = \begin{cases} 2x, & 0 \le x \le 1/2, \\ 2-2x, & 1/2 \le x \le 1, \end{cases}$$

for $t > 0$ as we considered in Section 4.2.

a) Implement the Crank-Nicolsons method on your computer. Since this is a voluntary exercise, use any language you like. If you choose to use Matlab, the code is given in Section 4.2.4. Observe time evolution of the solution. When time t is sufficiently large, how the solution look like?

[Solution] When time is sufficiently large, the solution converges to zero. Look at the first animation in the supplementary file.

b) The heat equation is known to have smoothing property: even if the initial condition is discontinuous (but in L_2), the solution at any $t > 0$ is in C^∞. Check this property numerically by choosing your own discontinuous initial condition.



[Solution] Look at the second animation in the supplementary file.

c) Modify your code for a general θ method, and compare the numerical solution (your choice of $\theta \neq 1/2$) with the Crank-Nicolsons method. Draw the convergence plot (in terms of M) of this method using as if the Crank-Nicolsons method is the analytic solution, where both methods should use the same number of points M. Theoretically, what convergence rate do you expect? And do you numerically see it?

[Solution][UPDATED]
Fix r to be some constant, say 1. For the Crank-Nicolson method we have the convergence rate $\mathcal{O}(k^2+h^2)=\mathcal{O}(h^4+h^2)=\mathcal{O}(h^2)$, whereas the theta method ($\theta\neq 1/2$) has the order $\mathcal{O}(k+h^2)=\mathcal{O}(h^2)$. Therefore, the difference between those two methods becomes $\mathcal{O}(h^2)=\mathcal{O}(M^{-2})$.

d) Consider a modified problem

$$u_t=-u_{xx},$$

with the same boundary/initial conditions. This is known to be an ill-posed problem where the solution diverges analytically. Consider the θ method to numerically solve this problem. Prove that you cannot obtain $F-$stability with any choice of parameters (Hint: look at Section 5.5).
[Solution] Assume that $0\leq\theta\leq 1$. Write the linear system in the following manner (look at Section 5.5):

$$(I+\theta rS)\boldsymbol{U}^{n+1}=((I+(\theta-1)rS)\boldsymbol{U}^{n}+\boldsymbol{d}^{n},$$

where S is the same as Section 5.5, and our boundary condition is expressed in $\boldsymbol{d}^n$. Note that because of the modified problem, the sign before the matrix S is now different. We want to prove that

$$\rho\left((I+\theta rS)^{-1}(I-(1-\theta)rS)\right)>1.$$

Now let

$$A:=(I+\theta rS)=\mathrm{tridiag}(r\theta,1-2r\theta,r\theta),$$
$$B:=(I-(1-\theta)rS)=\mathrm{tridiag}(-r(1-\theta),1+2r(1-\theta),-r(1-\theta)).$$

We know that (e.g., look at the previous exercise) for a tridiagonal matrix $\mathrm{tridiag}(c,a,b)$ the eigenvectors $\boldsymbol{x}^{(k)}$ and the associated eigenvalues λ_k are given by

$$x_j^{(k)}=\left(\frac{b}{c}\right)^{j/2}\sin\left(\frac{jk\pi}{M+1}\right),\quad \lambda_k=a+2\sqrt{bc}\cos\left(\frac{k\pi}{M+1}\right),$$

where $x_j^{(k)}$ is the jth element of the vector $\boldsymbol{x}^{(k)}$; $A\boldsymbol{x}^{(k)}=\lambda_k\boldsymbol{x}^{(k)}$. Observe that A and B have the same eigenvectors. This means we can diagonalize A and B by the same orthogonal matrix T:

$$A=T\Lambda_AT^{-1},B=T\Lambda_BT^{-1},$$

where Λ_A and Λ_B are diagonal matrices consisting of eigenvalues of A and B, respectively. Therefore,

$$(I+\theta rS)^{-1}(I-(1-\theta)rS)=A^{-1}B=T\Lambda_A^{-1}\Lambda_BT^{-1}.$$

So the eigenvalues of $A^{-1}B$ is

$$\frac{\lambda_{B,k}}{\lambda_{A,k}}=\frac{1+2(1-\theta)r+2(1-\theta)r\cos(k\pi/(M+1))}{1-2\theta r+2\theta r\cos(k\pi/(M+1))},$$

for $k=1,...,M$. With easy calculation one can show

$$\forall k\ \frac{\lambda_{B,k}}{\lambda_{A,k}}\geq 1,\ \text{and},\ \exists k\ \frac{\lambda_{B,k}}{\lambda_{A,k}}>1.$$

Therefore,

$$\rho(A^{-1}B)>1.$$

$\boxed{2}$ Solve the problems 3, 4 and 5 of the exercise 1 from 2020.

[Solution] Solution is given here.