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TMA4212 Numerical solution of differential equations by difference methods Spring 2021

Project 2

For this project 2D, you can ask a question on Piazza or to Yuya Suzuki (yuya.suzuki@ntnu.no).

D Consider the following wave equation on $x \in \mathbb{R}$, $t > 0$

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = g(x), \quad u_t(x,0) = h(x),$$

where c is a positive constant, with a periodic boundary condition

$$u(x+1,t) = u(x,t),$$

for all $x \in \mathbb{R}$ and $t > 0$. Note that due to this periodic boundary condition, we only need to consider one period $[0,1]$ for the spatial domain.

Consider equidistant points

$$x_0 = 0, \; x_1 = \frac{1}{M+1}, \dots, \; x_M = \frac{M}{M+1}, \; x_{M+1} = 1,$$

and let $h = 1/(M+1)$.

a) By using central differences on both directions x and t, we have

$$\frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{k^2} = c^2 \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2}, \tag{1}$$

where k is the time step size, for $(n+1)$th time step and $j = 0, 1, \dots, M$. When $n = 0$, we need to know U_j^{-1}. Express U_j^{-1} from the initial condition $u_t(x,0) = h(x)$ by using central difference in time, then rewrite (1) for $n = 0$ without using U_j^{-1}.

b) Now, by fixing $r = ck/h$ and by substituting in (1)

$$U_j^n = \xi^n \exp\left(2\pi \mathrm{i} \beta \frac{j}{M+1}\right),$$

where i is the imaginary unit and $j = 0, \dots, M$, derive the condition for r such that $|\xi| \le 1$ is satisfied for any $\beta \in \mathbb{Z}$.

Implement the difference method for such r with $c = 1$ and initial condition $g(x) = \cos(4\pi x)$, $h(x) = 0$. Derive the analytical solution for this problem by using separation of variables. Then make a convergence plot of the relative l_2 error in terms of h where $r = ck/h$ being fixed for the solution at time $t = 1$.

Change the initial condition to initial condition $g(x) = \exp(-100(x-1/2)^2)$, $h(x) = 0$. Observe the time development of the solution and describe it. Make some plots of the solution for different time t.

c) Add one more dimension to the problem: consider for $(x, y) \in \mathbb{R}^2$

$$u_{tt} = c^2(u_{xx} + u_{yy}), \;\; u(x, y, 0) = g(x, y), \;\; u_t(x, y, 0) = h(x, y),$$

with the periodic boundary condition on both spatial direction

$$u(x + 1, y, t) = u(x, y, t), \;\; u(x, y + 1, t) = u(x, y, t),$$

for all $(x, y) \in \mathbb{R}^2$ and $t > 0$. Again, due to this periodicity, we only need to consider one unit square $[0, 1]^2$ for the spatial domain. Consider equidistant points for both direction, but they can be different step sizes:

$$x_0 = 0, \; x_1 = \frac{1}{M+1}, ..., \; x_M = \frac{M}{M+1}, \; x_{M+1} = 1,$$

$$y_0 = 0, \; y_1 = \frac{1}{N+1}, ..., \; y_N = \frac{N}{N+1}, \; y_{N+1} = 1,$$

and let $h_x = 1/(M + 1)$, $h_y = 1/(N + 1)$. Generalize the difference method (1) for this 2D setting.

Again, by fixing $r_x = ck/h_x$, $r_y = ck/h_y$ and by substituting the following for the obtained 2D scheme

$$U^n_{j,l} = \xi^n \exp\left(2\pi \mathrm{i} \beta_1 \frac{j}{M+1}\right) \exp\left(2\pi \mathrm{i} \beta_2 \frac{l}{N+1}\right),$$

where $l = 0, ..., N$ and $j = 0, ..., M$, derive the condition for r_x and r_y such that $|\xi| \leq 1$ is satisfied for any $\beta_1, \beta_2 \in \mathbb{Z}$.

Implement the above difference method for $c = 1$ and the initial condition $g(x, y) = \cos(4\pi x) \sin(4\pi y)$, $h(x, y) = 0$. Derive the analytical solution for this problem by using separation of variables. Then make a convergence plot of the relative l_2 error in terms of h_x where $r_x = ck/h_x$ and $h_x = h_y$ being fixed, for the solution at time $t = 1$. Also, make some 3D plots (x, y, u) of the numerical solution for different time t.

d) For the above 2D problem, $c = 1$ and the initial condition $g(x, y) = \cos(4\pi x) \sin(4\pi y)$, $h(x, y) = 0$, and $t = 1$, to compare with other groups, submit a code which produces two graphs: x axis for the number of degree of freedom (MN/k) and y axis for the relative l_2 error; and x axis for the number of degree of freedom (MN/k) and y axis for computational time.