



1 Given the problem

$$\begin{aligned} -u_{xx} + u_x &= f(x), & 0 < x < 1, \\ u_x(0) + u(0) &= g_0, & u(1) = 0. \end{aligned}$$

Modify your code from Exercise 1, Problem 4, to handle this problem (only the boundary condition in $x = 0$ has changed.). Check the order of the approximation. You may try both first and second order approximations.

2 Consider the boundary value problem

$$-u_{xx} + u_x + u = f, \quad \text{on } [0, 1], \quad u_x(0) = u_x(1) = 0.$$

- a) Set up the weak formulation of the problem.
- b) Use Lax-Milgram theorem to prove that the problem has a unique solution.
- c) Given an equidistributed mesh $x_i = ih$, $h = 1/M$ on the interval $[0, 1]$ and the linear finite element space

$$X_h^1 = \{v \in C^0(0, 1); v|_{x_{i-1}, x_i} \in \mathbb{P}_1\}.$$

that is the space of continuous functions, that are also linear on each element $K_i = (x_i, x_{i+1})$. Notice that $X_h^1 \in H^1(0, 1)$.

Now use the Galerkin method to find an approximation to the problem.

- Set up the stiffness matrix A_h .
 - Find the load vector F_h .
 - Implement the method.
 - Test it on a freely chosen test problem.
- d) Find an error bound for the numerical solution, and verify it numerically. (This topic will be lectured on Tuesday 21.01).

3 Before doing this exercise, read section 2.3 in the note by Charles Curry.

Consider the problem

$$-u_{xx} = f, \quad u(0) = u(1) = 0.$$

This exercise is about how to solve this problem with a finite element method on the same equidistributed mesh as in the previous problem, but now with a finite element space

$$X_h^2 = \{v \in C^0(\Omega); v|_{x_{i-1}, x_i} \in \mathbb{P}_2\}.$$

The space X_h^2 can be constructed as follows: At each element (x_{i-1}, x_i) choose the points x_{i-1}, x_i and the midpoint $(x_{i-1} + x_i)/2$, and use the corresponding cardinal functions as basis functions.

- a) Let $M = 2$ and make a figure of the basis functions in this case.
- b) Set up the element stiffness matrix and the element load vector.
- c) Set up the complete stiffness matrix and load vector (by hand) for some arbitrary M . Use $f = 1$.

Solve the problem numerically for $M = 5$ (for example). If your calculations are correct, the scheme solves this particular problem exactly (why?)

Reminder: Given the gridpoints $(x_i)_{i=0}^n$, the corresponding cardinal-functions are given by

$$\ell_i(x) = \prod_{j=0, j \neq i}^n \frac{x - x_j}{x_i - x_j} \in \mathbb{P}_2$$

satisfying

$$\ell_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$