Norwegian University of Science and Technology
Department of Mathematical Sciences

Exercise set 1

1 Eigenvalues and eigenvectors of a tridiagonal matrix.

Given the $n \times n$ tridiagonal matrix

$$A = \begin{bmatrix} a & b & & & & \\ c & a & b & & & \\ & c & a & b & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & a & b \\ & & & & c & a \end{bmatrix} = \text{tridiag}\{c, a, b\} \in \mathbb{R}^{n\times n}.$$

Let $x^{(s)} = [x_1^{(s)}, \ldots, x_n^{(s)}]^\top$, and

$$x_k^{(s)} = \left(\frac{c}{b}\right)^{k/2} \sin k\phi_s, \qquad \lambda_s = a + 2\sqrt{bc}\cos\phi_s, \qquad \phi_s = \frac{s\pi}{n+1}.$$

a) Verify that $Ax^{(s)} = \lambda_s x^{(s)}$, $s = 1, \ldots, n$.
Hint: By simple insertion and uninhibited use of trigonometric equalities.

b) Find the eigenvalues to the matrix $A = \text{tridiag}\{-1, 2, -1\} \in \mathbb{R}^{n\times n}$.

2 Some useful difference approximations

a) Compute the error terms τ for the following approximations to derivatives of the function u in the point x_0:

$$\frac{u(x_0+h) - u(x_0)}{h} = u_x(x_0) + \tau \quad \text{(Forward difference approximation)}$$
$$\frac{u(x_0) - u(x_0-h)}{h} = u_x(x_0) + \tau \quad \text{(Backward difference approximation)}$$
$$\frac{u(x_0+h) - u(x_0-h)}{2h} = u_x(x_0) + \tau \quad \text{(Central difference approximation)}$$
$$\frac{u(x_0+h) - 2u(x_0) + u(x_0-h)}{h^2} = u_{xx}(x_0) + \tau \quad \text{(Central difference approximation).}$$

b) Construct a difference approximation to $u_x(x_0)$ based on $u(x_0)$, $u(x_0+h)$ and $u(x_0+2h)$ such that the error is $\mathcal{O}(h^2)$.

The next two exercises have already been done in the lectures, but they are given here as exercises to encourage you to work them through yourself. And yes, solutions will be provided for these too.

3 A discrete maximum principle

Prove the following statement:

Given the constants α, β and γ satisfying $\alpha, \beta, \gamma > 0$ and $\beta \geq \alpha + \gamma$. If a set $\{v_m\}_{m=1}^{M}$ satisfies the difference inequality

$$-\alpha v_{m-1} + \beta v_m - \gamma v_{m+1} \leq 0, \qquad m = 1, 2, \ldots, M-1,$$

then

$$\max_{m=0,\ldots,M} \{v_m\} = \max\{0, v_0, v_m\}.$$

In the lecture, we discussed the basic steps of developing and analysing a finite difference method for linear PDEs, with a two-point boundary value problem as an example. An outline for a procedure is given below, but for a given problem and scheme there may be all kinds of modifications required, so please take that into account.

1. Given a problem, that is a PDE (or BVP), the domain Ω and boundary conditions (BC).

$$Lu = f, \qquad x \in \Omega, \tag{1}$$
$$Bu = g, \qquad x \in \partial\Omega, \tag{2}$$

   where L is a linear differential operator, and B is the operator for the boundaries. Assume that all conditions for existence of a unique solution are satisfied.

2. Construct a finite difference discretisation of the problem:

   - Put a grid on the domain: $x_m \in \Omega$. In this course, we will mostly use a uniform grid, with a gridsize (stepsize) h.
   - Replace derivatives in L with appropriate difference approximations in each gridpoint x_m. Take the boundary conditions into account. The result should be a difference approximation, something like

$$\begin{aligned} L_h U_m &= f(x_m), &&\text{for all inner gridpoints } x_m, \\ B_h U_m &= g(x_m), &&\text{for all boundary gridpoints } x_m, \end{aligned} \tag{3}$$

     where $U_m \approx u(x_m)$ and in the following, we will write $u_m = u(x_m)$. (Modifications of this will happen, so do not take it too literary).

3. Implement the method, and test it, preferable on some test problem with a known solution.

4. Do an error analysis:

- Consistency: Find the truncation error τ_m, satisfying

$$
\begin{aligned}
L_h u_m &= f(x_m) + \tau_m, && \text{for all inner gridpoints } x_m, \\
B_h u_m &= g(x_m) + \tau_m, && \text{for all boundary gridpoints } x_m,
\end{aligned}
\tag{4}
$$

  You will find an expression for τ by Taylor-expansions around x_m. If, for all gridpoints x_m:
  - If $\tau_m \to_{h\to 0} 0$ the scheme is *consistent*
  - If $|\tau_m| \leq Ch^q$ for some constant C, the scheme is *consistent of order q*.

  The order of consistency is p if $\tau_m = \mathcal{O}(h^p)$, that is $|\tau_m\| \leq Ch^p$ for some constant p, for all the gridpoints.
- Convergence: The error is $e_m = u_m - U_m$. Since the operator L_h is linear, subtracting (3) from (4) results in the following relation between the global and the local error.

$$
\begin{aligned}
L_h e_m &= \tau_m, && \text{for all inner gridpoints } x_m, \\
B_h e_m &= \tau_m, && \text{for all boundary gridpoints } x_m,
\end{aligned}
\tag{5}
$$

  The next step is to prove some bound for e_m given that a bound for τ_m is known, that is if $|\tau_m| \leq Ch^q$ then $|e_m| < \tilde{C}h^p$ (or similar) for all gridpoints. This is probably the hardest part of the analysis, and good strategies depends on the problem and schemes available.

- If $e_m \to_{h\to 0} 0$ the scheme is *convergent.*
- If $|e_m| \leq Ch^p$ for some constant C, the scheme is *convergent of order p.*

5. Verify the numerical results, typically by a convergence plot.
   - If possible, choose a (not too trivial) test problem with a known exact solution.
   - Solve the problem for different stepsizes h_i, and for each stepize, measure the corresponding error $E(h_i)$ (for instance $E = max_m|e_m|$). Use a loglog plot, since, if

$$
E(h) \approx Ch^p \qquad \Rightarrow \qquad \log E(h) = p\log(h) + \log(C)
$$

     $\log E(h)$ is a straight line with slope p. This can also be used to approximate the order p from the measurements (use the function `polyfit`).

4 Given the problem:

$$
\begin{aligned}
-u_{xx} + u_x = f(x), \qquad 0 < x < 1, \\
u(0) = g_0, \qquad u(1) = g_1,
\end{aligned}
$$

with $c \geq 0$.

Do the whole procedure outlined above on this problem.

Hint: For the error analysis: Apply the discrete maximum principle on the set v_m, where $v_m = \phi_m \pm e_m$, with $\phi(x) = 2Dx(x-1)$ to get a bound for $|e_m|$.

The solution to the next problem will be provided together with solutions of Exercise 2.

5 Given the problem

$$-\mu u_{xx} + b u_x = f(x), \qquad 0 < x < 1,$$
$$u(0) = g_0, \qquad u(1) = g_1,$$

where μ and b are constants, and $\mu > 0$.

Do the whole procedure outlined above for this problem. Use second order approximations to the derivatives.

When you have a convergence result, rewrite it as a theorem with a proper proof.

Hint: You are allowed to set some upper bounds on the gridsize to make the discrete maximum principle to work.