NTNU
Norwegian University of
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Department of Mathematical Sciences

Examination paper for
TMA4212 Numerical Solution of Partial Differential Equations By Difference Methods

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Examination time (from–to): 09:00–13:00

Permitted examination support material: C: Approved simple pocket calculator is allowed. The text book by Strikwerda, the book by Süli and Mayers, and the two official notes of the TMA4212 course (98 pages and 28 pages) are allowed. Rottman is allowed. The books in printed version are also allowed. Old exams with solutions are not allowed.

Language: English

Number of pages: 2

Number of pages enclosed: 2

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
sort/hvit ☒	**farger** ☐
skal ha flervalgskjema ☐	

Date Signature

The learning outcomes have been published on the course webpage and in the official description of the course. The seven learning goals **L1** to **L7** are reported in the appendix. Learning outcomes **L3**, **L4** and **L6** have been tested through the project work. Here **L4** is tested further, together with outcomes **L1, L2, L5, L7**.

All answers must be properly argued for.

Problem 1 (L1, L4, L7)

Consider the equation

$$u_t = t u_x, \quad 0 < x < 1$$

a) Show that the characteristics of the above equation take the form

$$x(t) = x_0 - \frac{1}{2}t^2$$

The equation requires a boundary condition to be well-posed. Give an example Dirichlet boundary condition and solve the resulting problem explicitly using the method of characteristics.

b) Which of the following methods would you use to approximate the solution of the equation?

$$U_m^{n+1} = U_m^n - \frac{nk}{h}(U_{m+1}^n - U_m^n)$$

$$U_m^{n+1} = U_m^n - \frac{nk}{h}(U_m^n - U_{m-1}^n)$$

c) Suppose you employ your choice of the above methods on a uniform grid with $h = \frac{1}{4}$. What restriction is required on k if the method is to satisfy the CFL condition at all times $t \leq 2$?

Problem 2 (L1, L4)

The diffusion equation

$$u_t = u_{xx}, \quad u(x,0) = f(x)$$

is discretized as follows:

$$U_m^{n+1} - \frac{1}{2}(\frac{k}{h^2} - \frac{1}{6})(U_{m-1}^{n+1} - 2U_m^{n+1} + U_{m+1}^{n+1}) = U_m^n + \frac{1}{2}(\frac{k}{h^2} + \frac{1}{6})(U_{m-1}^n - 2U_m^n + U_{m+1}^n),$$

where $U_m^n = U(mh, nk)$ is the approximate solution on a grid of step sizes h in space and k in time.

a) Find the leading error term of the local truncation error of this method.

b) Assume that method is simulated under periodic boundary conditions. Perform a Von Neumann stability analysis.

Problem 3 (L2, L7)

We consider the boundary value problem

$$-\frac{d}{dx}\left(\left(x+\frac{1}{3}\right)\frac{du}{dx}\right)+9u=0,\quad 0<x<1,\quad u(0)=1,u(1)=-1.$$

a) Let R be an arbitrary lifting of the boundary conditions, such that $\hat{u}=u-R$ solves a homogeneous Dirichlet problem. Find a variational form of the equation for $\hat{u}$.

b) Suppose we approximate the solution of the given equation by the finite element method on a uniform grid

$$0=x_0<x_1<\ldots<x_M<x_{M+1}=1,$$

using linear nodal basis functions. Show that this results in a linear system

$$A\hat{u}=b,$$

and argue that A is positive definite, and hence that there exists a unique solution to the discrete problem (you should not calculate the entries of A or b explicitly at this stage, simply indicate how they can be found).

c) Suppose $M=2$, such that the grid is $x_0=0, x_1=\frac{1}{3}, x_2=\frac{2}{3}, x_3=1$. Find the resulting approximate solution u.

Problem 4 (L5)

Let $\alpha,\beta\in\mathbb{R}$, and consider the linear system

$$\begin{pmatrix}\alpha & \beta & 0\\ \beta & \alpha & \beta\\ 0 & \beta & \alpha\end{pmatrix}u=\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

a) Perform two steps of the Jacobi iteration starting from $u_0=(0,0,0)^T$.

b) For what values of α,β does the Jacobi iteration converge?

Appendix

- **Jacobi iteration**: given the linear system of equations

  $$Ax = b$$

  with A $n \times n$ matrix and b a vector with n components, we split A as the sum of its diagonal D minus a matrix R:

  $$A = D - R.$$

  We assume that D is invertible.

  The Jacobi iteration is an iterative method to approximate the solution of the linear system, and is given by the iteration

  $$x^{k+1} = D^{-1}(R\,x^k + b), \tag{1}$$

  with x^0 a given initial guess. Note that (1) this is a fixed point iteration to solve the fixed point equation $x = D^{-1}(R\,x + b)$, whose solution is the same as for the linear system.

Learning outcome:

Knowledge	**L1**	Understanding of error analysis of difference methods: consistency, stability, convergence of difference schemes.
	L2	Understanding of the basics of the finite element method.
Skills	**L3**	Ability to choose and implement a suitable discretization scheme given a particular PDE, and to design numerical tests in order to verify the correctness of the code and the order of the method.
	L4	Ability to analyze the chosen discretization scheme, at least for simple PDE-test problems.
	L5	Ability to attack the numerical linear algebra challenges arising in the numerical solution of PDEs.
General competence	**L6**	Ability to present in oral and written form the numerical and analytical results obtained in the project work.
	L7	Ability to apply acquired mathematical knowledge in linear algebra and calculus to achieve the other goals of the course.