Problem 1:

a) Scheme:

$$\frac{1}{k}(U_i^{n+1} - U_i^n) = \frac{a}{h^2}(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}) - U_i^{n+1},$$

$$\Rightarrow \boxed{\begin{aligned} &U_i^{n+1} = U_i^n + a\frac{k}{h^2}(U_{i+1}^{n+1} - 2U_i^{n+1} + U_i^{n+1}) - k\cdot U_i^{n+1}, \quad && i = 1, \cdots, M-1 \\ & && n = 0, 1, 2, \cdots \\ &U_0^n = U_M^n = 0 \end{aligned}}$$

Local truncation error, τ_i^n: Let $u_i^n = u(x_i, t_n)$

$$k\cdot\tau_i^n = u_i^{n+1} - u_i^n - a\frac{k}{h^2}(u_i^{n+1} - 2u_i^{n+1} + u_i^{n+1}) - k u_i^{n+1}$$

For simplicity, do the expansion around (x_i, t_{n+1}):

$$k\cdot\tau_i^n = k\cdot\partial_t u + \tfrac{1}{2}k^2\partial_t^2 u + \mathcal{O}(k^3)$$

$$- a\cdot k[\partial_x^2 u + h^2\cdot\tfrac{1}{12}\partial_x^4 u + \mathcal{O}(h^6)] + k\cdot u$$

so, using that $\partial_t u - \partial_x^2 u + u = 0$ we get

$$\boxed{\tau_i^n = \tfrac{1}{2}k\,\partial_t^2 u - h^2\frac{a}{12}\partial_x^4 u + \mathcal{O}(k^2) + \mathcal{O}(h^4)}$$

all evaluated in (x_i, t_{n+1}), or in (x_i, t_n).
The difference will only appear in the higher order terms.

b) The scheme is consistent ($\tau_i^n \xrightarrow[h,k\to 0]{} 0$), so according to the Lax equivalence theorem, it is sufficient to prove stability.

Let $r = k/h^2$. The scheme can be written as

$$U_i^{n+1} - r(U_i^{n+1} - 2U_i^{n+1} + U_i^{n+1}) + k\cdot U_i^{n+1} = U_i^n.$$

Using $U^n = [U_1^n, \cdots, U_{M-1}^n]^T$. this becomes

$$A\cdot U^{n+1} = U^n \Rightarrow U^{n+1} = C\cdot U^n \quad \text{with } C = A^{-1}, \text{ and}$$

where $A = (1+k)I + r\cdot T$, $T = \text{tridiag}\{-1, 2, -1\}$

We know: The scheme is stable if $\rho(C) \leq 1$

The eigenvalues are:

$$\lambda_j(T) = 4\sin^2\left(\frac{j\pi}{2M}\right) \qquad \text{(from the note)}$$

$$\lambda_j(A) = (1+k) + r \cdot \lambda_j(T) = 1 + k + 4r\sin^2\left(\frac{j\pi}{2M}\right)$$

and

$$\lambda_j(C) = \frac{1}{\lambda_j(A)} = \frac{1}{1+k+4r\sin^2\left(\frac{j\pi}{2M}\right)}$$

and clearly

$$0 < \lambda_j(C) < 1 \quad \Rightarrow \quad \rho(C) < 1$$

and the scheme is stable, and thus convergent.

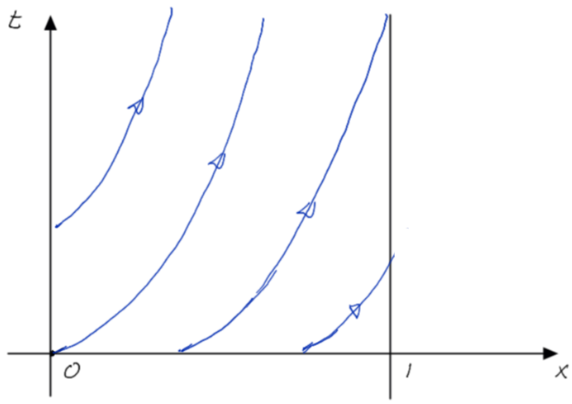
Problem 2:

a) The characteristics are the solution of the equations

$$\frac{dx}{dt} = \frac{1}{(1+t)^2} \qquad x(t^*) = x^*.$$

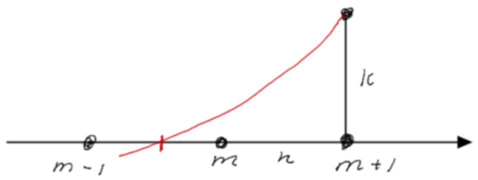
The solution is

$$\boxed{x(t) = x^* + \frac{1}{1+t^*} - \frac{1}{1+t}}$$



The characteristics are moving from left to right, so we should include boundary conditions at $x = 0$.

b) We notice that the velocity $\frac{1}{(1+t)^2}$ becomes smaller as t increases, so the for the CFL condition to be satisfied everywhere, it suffice to have it satisfied for the first step. The requirement then is that the characteristic through (x_{m+1}, k) should pass through the interval (x_{m-1}, x_{m+1}) at $t=0$ (se the figure).



Computational stencil

The condition becomes:

$$x(0) = x_{m+1} + \frac{1}{1+k} - 1 \in [x_{m-1}, x_{m+1}]$$

$$\Rightarrow x_{m+1} + \frac{1}{1+k} - 1 \geq x_{m-1}$$

$$\Rightarrow 2h + \frac{1}{1+k} \geq 1 \Rightarrow \boxed{k \leq \frac{2h}{1-2h}}$$

If $h = 1/4$, $k \leq 1$.

Problem 3:

This is not a complete solution to this problem, but some indications on what can be done:

If we do not know much about v and a, it may be a good idea to go for 2. order approximations;

$$v(u)u_x \quad \rightarrow \quad v(U_i)\cdot\frac{1}{2h}\left(U_{i+1} - U_{i-1}\right)$$

$$(a(u)u_x)_x \quad \rightarrow \frac{1}{h^2}\left(a_{i+1/2}\,U_{i+1} - (a_{i+1/2} + a_{i-1/2})U_i + a_{i-1/2}\,U_i\right)$$

where one of the following can be used;

$$a_{i+1/2} = \frac{1}{2}\left(a(U_{i+1}) + a(U_i)\right)$$

or

$$a_{i+1/2} = a\left(\frac{1}{2}(U_{i+1} + U_i)\right)$$

Boundary conditions:

$x = 0$: $\quad U_0 = 0$

$x = 1$: $\quad u_x + u = 0 \quad \rightarrow \quad \dfrac{U_{M+1} - U_{M-1}}{2h} + U_M = 0$

$$\Rightarrow U_{M+1} = U_{M-1} - 2h\,U_M$$

which is inserted into the difference formula (not written out here) for $i = M$.

Altogether, this ends up in a system of ODEs.

$$\dot{U} = F(U) \quad , \text{ where } \quad U = [U_1, \dots, U_M].$$

Since we have a diffusion term, we would choose an implicit (A-stable) scheme to solve the ODE. The trapezoidal rule (Crank-Nicolson) is of order 2, so that may be a good choice:

$$U^{n+1} = U^n + \frac{h}{2}\left[F(U^{n+1}) + F(U^n)\right].$$

which altogether creates a system of nonlinear equations which has to be solved for each step.

Other considerations:

If the diffusion term is small, you may want to use an upwind scheme ($v > 0$):

$$v(u)u_x \quad \rightarrow \quad v(U_i)\,\frac{U_i - U_{i-1}}{h}. \qquad \text{(order 1)}$$

In this case, it may also be possible to use
an explicit scheme, but stability may be an issue.

Problem 4: $\quad a > 0$

Scheme: $U_m^{n+1} = U_m^n - p \cdot (U_m^n - U_{m-1}^n)$

Let $U_m^n = \xi^n e^{i\beta x_m}$, and we get

$$\xi = 1 - p(1 - e^{-i\beta h}) = 1 - p + p(\cos(\beta h) - i\sin(\beta h))$$

and

$$|\xi|^2 = (1 - p + p\cos(\beta h))^2 + p^2 \sin^2(\beta h)$$

$$= 1 - 2p(1-p)(1 - \cos\beta h) \geq 0 \quad , \quad -\pi < \beta h \leq \pi$$

So, for $p = 1$, $|\xi|^2 = 1$ and there are no dissipation

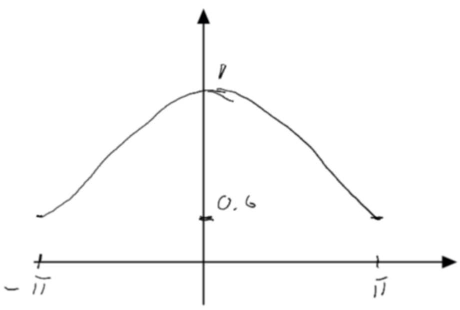
$p > 1$ or $p < 0$: $|\xi|^2 > 0$ for some βh, and the scheme is unstable. This coincide with the CFL-condition.

$0 < p < 1$: In this case, $|\xi| \leq 1$ and the scheme is stable. ξ depends on βh, so the method is dissipative. It can be shown that

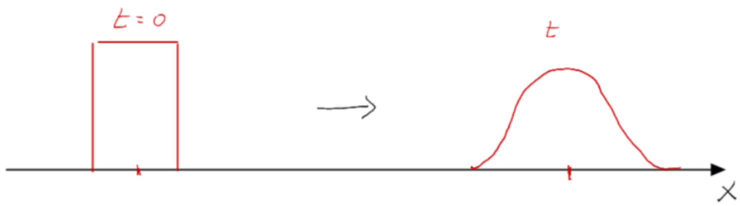
$$|\xi| = 1 - \tfrac{1}{2} p(p-1)(\beta h)^2 + O((\beta h)^4)$$

thus the method is dissipative of order 2.

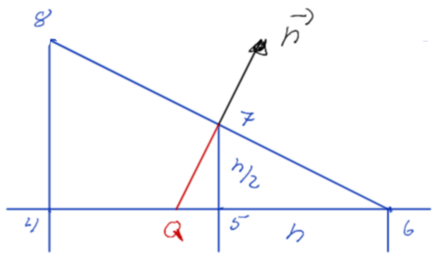
For $p = 0.8$, $\xi = \sqrt{0.68 + 0.3 \cdot \cos(\beta h)}$, is given by



and high frequeny modes will be damped faster than slow modes. Consequently, the the solution will be smoothed out over time.



Problem 5:



An approximation to the solution in point Q can be found by linear intepolation:

The distance from Q to node 5 is

$$h' = h/4.$$

$$u_Q \approx u_4 + \frac{u_5 - u_4}{h} \cdot \frac{3}{4}h$$

$$= \frac{3}{4}u_5 + \frac{1}{4}u_4$$

The distance from node 7 to Q is $\frac{\sqrt{5}}{4}h$, so a suitable approximation to $\partial_n u$ in 7 may be

$$\frac{U_7 - U_Q}{\frac{\sqrt{5}}{4}h} = g_7 \Rightarrow \frac{4}{\sqrt{5}h}\left(U_7 - \frac{3}{4}U_5 - \frac{1}{4}U_4\right) = g_7$$

$$\Rightarrow \boxed{\frac{1}{\sqrt{5}h}\left(4U_7 - 3U_5 - U_4\right) = g_7}$$

To find the local truncation error, notice that the unit normal vector $\vec{n}$ is

$$\vec{n} = [1/\sqrt{5}, \ 2/\sqrt{5}]^T \quad \text{so}$$

$$\partial_n u = \vec{n} \cdot \nabla u = \frac{1}{\sqrt{5}}(u_x + 2u_y).$$

Then

$$\tau_7 = \frac{1}{\sqrt{5}h}\left(4u(x_7, y_7) - 3u(x_7, y_7 - \tfrac{h}{2}) - u(x_7 - h, y_7 - \tfrac{h}{2})\right) - g_7$$

$$= \frac{1}{\sqrt{5}h}\Big(4u - 3(u - \tfrac{h}{2}u_y + \tfrac{1}{2}(\tfrac{h}{2})^2 u_{yy} + O(h^3))$$

$$- (u - hu_x - \tfrac{h}{2}u_y + \tfrac{1}{2}h^2 u_{xx} + \tfrac{1}{2}h^2 u_{xy} + \tfrac{1}{2}(\tfrac{h}{2})^2 u_{yy} + O(h^3))\Big)$$

$$- g_7$$

$$= \overbrace{\frac{1}{\sqrt{5}}(u_x + 2u_y) - g_7}^{=0} - \frac{h}{2}(u_{xx} + u_{yy} + u_{xy}) + O(h^2)$$

so the local truncation error is of order 1.

Problem 6.

$$-(a(x)\cdot u_x)_x = 1 \qquad 0<x<1$$
$$u_x(0)=1 \quad , \quad u(1)=0.$$

a) Multiply by a function v, use partial integration to get

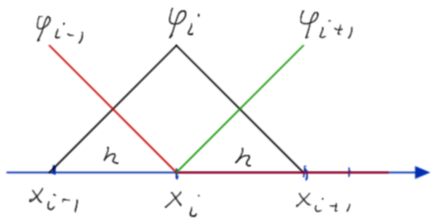
$$\int_0^1 a u_x v_x \, dx - \Big|_0^1 u_x v = \int_0^1 v \, dx$$

For the integrals to exist, $u, v \in H^1(\Omega)$, $\Omega = (0,1)$. We know that $u_x(0)=1$, To get rid of the boundary term at $x=1$, we require $v(0)=0$ (and we know that $u(0)=0$).

Let $V = H^1_r(\Omega) = \{ v \in H^1(\Omega), \ v(1)=0 \}$ $(\Omega = (0,1))$.

The weak formulation is now:

$$\boxed{\text{Find } u \in H^1_r \text{ s.t. } \int_0^1 a u_x v_x \, dx = \int_0^1 v \, dx - v(1), \quad \forall v \in H^1_r(\Omega)}$$

b) 

The linear nodal basis functions are described in the figure.

$$\varphi_i' = \begin{cases} 1/h & x_{i-1} < x < x_i \\ -1/h & x_i \le x < x_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

The elements of the stiffness matrix is

$$A_{ij} = \int_0^1 (x + 1/2)\varphi_i' \varphi_j' \, dx \quad , \quad i = 0, \dots, M-1$$

The matrix is tridiagonal, with elements: , $i = 1, \dots, M-1$

$$A_{ii} = \frac{1}{h^2}\int_{x_{i-1}}^{x_{i+1}} (x + \tfrac{1}{2}) \, dx = \frac{1}{h}(2x_i + 1)$$

$$A_{i,i+1} = -\frac{1}{h^2}\int_{x_i}^{x_{i+1}} (x + \tfrac{1}{2}) \, dx = -\frac{1}{h}\left(x_i + \tfrac{1}{2}(1+h)\right)$$

$$A_{i,i-1} = -\frac{1}{h}\left(x_i + \tfrac{1}{2}(1-h)\right)$$

For the boundary function φ_0, we get.

$$A_{00} = \frac{1}{h^2}\int_0^h (x + \tfrac{1}{2}) \, dx = \frac{1}{2h}(1+h)$$

$$A_{01} = -\frac{1}{h^2}\int_0^h (x + \tfrac{1}{2}) \, dx = -\frac{1}{2h}(1+h)$$

For the load vector $\vec{b}$, all elements but the first is

$$b_i = \int_{x_{i-1}}^{x_{i+1}} 1 \cdot \varphi_i(x)\,dx = 2 \cdot \frac{1}{2} h = h$$

For the first one we get

$$b_0 = \int_0^h 1 \cdot \varphi_0(x)\,dx + \varphi_0(1) = \frac{h}{2} + 1$$