NTNU – Trondheim
Norwegian University of
Science and Technology

Department of Mathematical Sciences

# Examination paper for
# **TMA4212 Numerical solution of differential equations with difference methods**

**Academic contact during examination:** Sølve Eidnes

**Phone:**

**Examination date:** 22. May 2017

**Examination time (from–to):** 09:00-13:00

**Permitted examination support material:** C: Approved simple pocket calculator is allowed. The text book by *Strikwerda*, the book by *Süli and Mayers*, and the official note of the TMA4212 course (98 pages) are allowed. Photo copies on 2D finite elements (4 pages) are allowed. Rottman is allowed. The books in printed version are also allowed. Old exams with solutions are *not* allowed.

**Language:** English

**Number of pages:** 6

**Number of pages enclosed:** 2

**Checked by:**

Date Signature

The learning outcome has been published on the course webpage and on the official description of the course. The seven learning goals **L1** to **L7** are reported in the appendix. Learning outcome **L6**, **L3** and **L4** have been tested through the project work. We here test the achievement of **L1, L2, L5 L7**.

All answers must be properly argued for.

**Problem 1** (**L2**)

Consider the boundary value problem

$$\begin{aligned} -u'' + u &= f(x), \quad 0 < x < 1 \\ & u(0) = 0, \quad u(1) + \beta u'(1) = \gamma, \end{aligned}$$

$f \in L^2(0,1)$, with $\beta \neq 0$.

**a)** State the Galerkin formulation of the problem, identify the space of test functions, the bilinear form and the linear form.

**Solution**: The space of test functions is

$$S := \{v \in H^1(0,1) \,|\, v(0) = 0\};$$

the bilinear form is

$$A(u,v) := \int_0^1 (u'v' + uv)\,dx + \frac{1}{\beta} u(1)v(1);$$

the linear form is

$$\ell(v) := \int_0^1 f v\,dx + \frac{\gamma}{\beta} v(1).$$

The Galerkin formulation becomes:
*Find* $u \in V$ *satisfying*

$$A(u,v) = \ell(v), \qquad \forall v \in V.$$

**b)** Assume $\beta = 0$ and $\gamma = 0$ and modify the Galerkin formulation appropriately. Consider a general nonuniform subdivision of $[0,1]$ with points

$$0 = x_0 < x_1, \ldots, x_{N-1} < x_N = 1,$$

where the mesh-points $x_i$ are not necessarily equally spaced. Let $h_i = x_i - x_{i-1}$ and let $h = \max_i h_i$. Consider the finite element basis functions

$$\phi_i(x) = \begin{cases} 0 & \text{if} \quad x \le x_{i-1} \\ \frac{x - x_{i-1}}{h_i} & \text{if} \quad x_{i-1} \le x \le x_i \\ \frac{x_{i+1} - x}{h_{i+1}} & \text{if} \quad x_i \le x \le x_{i+1} \\ 0 & \text{if} \quad x_{i+1} \le x, \end{cases}$$

for $i = 1, \dots, N-1$. Formulate the Galerkin method. Consider $N = 3$ and $x_0 = 0$, $x_1 = 1/4$, $x_2 = 3/4$ and $x_3 = 1$. Specify the function space $S_0^h$ by writing explicit expressions of the basis functions $\phi_i$.

**Solution**: The Galerkin formulation is

*Find* $u \in S$ *satisfying* $A(u, v) = \ell(v), \qquad \forall v \in S$

with

$$S := H_0^1(0,1) = \{v \in H^1(0,1) \,|\, v(0) = 0, \quad v(1) = 0\},$$

and

$$A(u, v) := \int_0^1 (u'v' + uv)\, dx, \quad \ell(v) := \int_0^1 fv\, dx.$$

Defining

$$S_0^h := \left\{ v^h \in H_0^1(0,1) \,|\, v^h(x) = \sum_{i=1}^{N-1} v_i \phi_i(x) \right\}$$

the Galerkin method is formulated as

*Find* $u^h \in S_0^h$ *such that* $A(u^h, v^h) = \ell(v^h)$ *for all* $v^h \in S_0^h$.

For $N = 3$ there are two basis functions. These are

$$\phi_1(x) = \begin{cases} 4x & \text{if} \quad 0 \le x \le \frac{1}{4} \\ \frac{3}{2} - 2x & \text{if} \quad \frac{1}{4} \le x \le \frac{3}{4} \\ 0 & \text{if} \quad \frac{3}{4} \le x \le 1 \end{cases} \qquad \phi_2(x) = \begin{cases} 0 & \text{if} \quad 0 \le x \le \frac{1}{4} \\ 2x - \frac{1}{2} & \text{if} \quad \frac{1}{4} \le x \le \frac{3}{4} \\ 4 - 4x & \text{if} \quad \frac{3}{4} \le x \le 1 \end{cases}$$

To find the vector $\mathbf{u}$ whose components $u_i$ identify the numerical approximation

$$u^h(x) = \sum_{i=1}^{N-1} u_i \phi_i(x),$$

we need to solve a linear system of algebraic equations

$$A\,\mathbf{u} = \mathbf{b}.$$

**c)** Find the entries of the matrix $A$ of the linear system of algebraic equations, in the case $N = 3$ and $x_0 = 0$, $x_1 = 1/4$, $x_2 = 3/4$ and $x_3 = 1$.

**Solution:** The entires of the $2 \times 2$ matrix of the linear system are

$$A(\phi_1, \phi_2) = \frac{-23}{12}, \quad A(\phi_1, \phi_1) = \frac{75}{12}, \quad A(\phi_2, \phi_2) = \frac{75}{12}.$$

Suppose that the solution of the boundary value problem $u \in H_0^2(0,1)$. Consider the function

$$\mathcal{I}_h u = \sum_{i=1}^{N-1} u(x_i)\phi_i(x),$$

which is called the interpolant of $u$. Notice that $\mathcal{I}_h u \in S_0^h$. It is possible to show that the following estimate holds

$$\|u - \mathcal{I}_h u\|_{H_0^1(0,1)} \leq \frac{h}{\pi}\left(1 + \frac{h^2}{\pi^2}\right)^{\frac{1}{2}} \|u''\|_{L^2(0,1)}. \tag{1}$$

**d)** Using Céa's lemma and (1), show that the same bound holds for the error in the $H^1(0,1)$ norm, i.e.

$$\|u - u_h\|_{H^1(0,1)} \leq \frac{h}{\pi}\left(1 + \frac{h^2}{\pi^2}\right)^{\frac{1}{2}} \|u''\|_{L^2(0,1)}.$$

**Solution**: We observe that for $v \in H^1(0,1)$, $\|v\|_{H^1(0,1)} = A(v,v)^{\frac{1}{2}}$. Consider the error in the $H^1$ norm by Céa's lemma we have

$$\|u - u_h\|^2_{H^1(0,1)} = A(u - u_h, u - u_h) \leq A(u - v_h, u - v_h), \quad \forall v_h \in S_0^h$$

so

$$\|u - u_h\|_{H^1(0,1)} \leq A(u - \mathcal{I}_h u, u - \mathcal{I}_h u)^{\frac{1}{2}} = \|u - \mathcal{I}_h u\|_{H^1(0,1)} \leq \frac{h}{\pi}\left(1 + \frac{h^2}{\pi^2}\right)^{\frac{1}{2}} \|u''\|_{L^2(0,1)}.$$

**Problem 2** (**L1, L7**)

The eigenvalue problem

$$-y'' = \lambda y, \quad y(0) = y(1) = 0,$$

is approximated by

$$-\frac{Y_{j+1} - 2Y_j + Y_{j-1}}{h^2} = \mu Y_j, \quad 1 \le j \le n-1, \quad Y_0 = Y_n = 0.$$

**a)** The solution of the differential equation is $y = \sin(m\,\pi\,x)$, $\lambda = m^2\,\pi^2$ for any positive integer $m$. Show that the difference approximation has solution $Y_j = \sin(m\,\pi\,x_j)$, $j = 0, 1, \ldots, n$, and give an expression for the corresponding value of $\mu$.

**Solution**: Inserting the suggested expression for the solution of the difference equation we obtain

$$-\frac{\sin(m\pi(x_j - h)) - 2\sin(m\pi x_j) + \sin(m\pi(x_j + h))}{h^2} = \mu \sin(m\pi x_j).$$

Using trigonometric identities we obtain

$$\sin(m\pi(x_j - h)) + \sin(m\pi(x_j + h)) = 2\sin(m\pi x_j)\cos(m\pi h),$$

which inserted in the equation leads to

$$[2 - h^2\mu - 2\cos(m\pi h)]\sin(m\pi x_j) = 0$$

so

$$\mu = \frac{2}{h^2}(1 - \cos(m\pi h)).$$

**b)** Use the fact that

$$1 - \cos\theta = \frac{1}{2}\theta^2 - \frac{1}{24}\xi\theta^4, \quad |\xi| \le 1,$$

to show that $|\lambda - \mu| \le m^4\pi^4h^2/12$.

**Solution**: Using the suggested expansion we get

$$\mu = \frac{2}{h^2}\left(\frac{1}{2}(m\pi h)^2 - \frac{1}{24}\xi(m\pi h)^4\right), \quad |\xi| < 1,$$

so

$$|\lambda - \mu| = |m^2\pi^2 - m^2\pi^2 + \frac{1}{12}\xi m^4\pi^4h^2| \le \frac{1}{12}m^4\pi^4h^2.$$

**c)** Consider the truncation error

$$\tau_j := -\frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} - \lambda y_j, \quad j = 1, \ldots, n-1, \quad y_j = y(x_j).$$

Show that

$$\tau_j = -\frac{1}{12} h^2 m^4 \pi^4 \sin(m\,\pi\,\xi_j), \quad \xi_j \in (x_{j-1}, x_{j+1}).$$

**Solution**: By Taylor expansion we have

$$\tau_j = -\frac{1}{12} h^2 y^{(4)}(\xi_j), \quad \xi_j \in (x_{j-1}, x_{j+1}),$$

and using the knowledge of the solution $y = \sin(m\,\pi\,x)$ we obtain easily that $y^{iv} = y$, so

$$\tau_j = -\frac{1}{12} h^2 m^4 \pi^4\, y(\xi_j) = -\frac{1}{12} h^2 m^4 \pi^4 \sin(m\,\pi\,\xi_j), \quad \xi_j \in (x_{j-1}, x_{j+1}).$$

**d)** From **a)**, we notice that for this problem $Y_j = y(x_j) = y_j$. Assuming

$$\begin{aligned} \mathbf{Y} &= (Y_1, \ldots, Y_{n-1})^T, \\ \mathbf{y} &= (y_1, \ldots, y_{n-1})^T, \\ \boldsymbol{\tau} &= (\tau_1, \ldots, \tau_{n-1})^T, \end{aligned}$$

the equations can now be written in the form

$$(A - \mu I)\mathbf{Y} = 0,$$

with $A$ the $(n-1) \times (n-1)$ matrix arising by the central difference discretisation of the second derivative. The truncation error satisfies

$$(A - \lambda I)\mathbf{y} = \boldsymbol{\tau}.$$

Using the last two equations and the result in **c)** obtain an estimate of the error

$$|\lambda - \mu|.$$

**Solution**: Subtracting the two equations we obtain

$$(\mu - \lambda)\mathbf{y} = \boldsymbol{\tau},$$

which component wise is

$$(\mu - \lambda) \sin(m\pi x_j) = -\frac{1}{12} h^2 m^4 \pi^4 \sin(m\pi\xi_j), \quad j = 1, \ldots, n-1.$$

So

$$|\mu - \lambda| \leq \frac{1}{12} h^2 m^4 \pi^4 \left| \frac{\sin(m\pi\xi_j)}{\sin(m\pi x_j)} \right|, \quad j = 1, \ldots, n-1.$$

**Problem 3** (**L5, L7**) Assume $a \in \mathbf{R}$ and consider the matrix

$$A = \begin{bmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{bmatrix}.$$

**a)** For which values of $a$ is $A$ positive definite?

**Solution**: The eigenvalues of $A - I$ are $\{2a, -a, -a\}$ and the eigenvalues of $A$ are $\{2a + 1, 1 - a, 1 - a\}$. So the eigenvalues of $A$ are positive when $-\frac{1}{2} < a < 1$. So for these values of $a$, $A$ is positive definite.

**b)** Consider the Jacobi iteration for solving a linear system $Ax = b$, see the appendix. For which values of $a$ does the Jacobi method converge?

**Solution**: The iteration matrix of the Jacobi method is $I - A$ whose eigenvalues are $\{-2a, a, a\}$. The spectral radius

$$\rho(I - A) = \max\{|2a|, |a|\} = 2\,|a|$$

and

$$\rho(I - A) < 1 \Leftrightarrow |a| < \frac{1}{2}.$$

**Appendix**

- $$\frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2} = u''(x_i) + \frac{h^2}{12}u''''(x_i) + \mathcal{O}(h^4)$$

- **Jacobi iteration**: given the linear system of equations

  $$Ax = b$$

  with $A$ $n \times n$ matrix and $b$ a vector with $n$ components, we split $A$ as the sum of its diagonal $D$ minus a matrix $R$:

  $$A = D - R.$$

  We assume that $D$ is invertible.

  The Jacobi iteration is an iterative method to approximate the solution of the linear system, and is given by the iteration

  $$x^{k+1} = D^{-1}(R\,x^k + b), \tag{2}$$

  with $x^0$ a given initial guess. Note that (2) this is a fixed point iteration to solve the fixed point equation $x = D^{-1}(R\,x + b)$, whose solution is the same as for the linear system.

**Learning outcome:**

| | | |
|---|---|---|
| Knowledge | **L1** | Understanding of error analysis of difference methods: consistency, stability, convergence of difference schemes. |
| | **L2** | Understanding of the basics of the finite element method. |
| Skills | **L3** | Ability to choose and implement a suitable discretization scheme given a particular PDE, and to design numerical tests in order to verify the correctness of the code and the order of the method. |
| | **L4** | Ability to analyze the chosen discretization scheme, at least for simple PDE-test problems. |
| | **L5** | Ability to attack the numerical linear algebra challenges arising in the numerical solution of PDEs. |
| General competence | **L6** | Ability to present in oral and written form the numerical and analytical results obtained in the project work. |
| | **L7** | Ability to apply acquired mathematical knowledge in linear algebra and calculus to achieve the other goals of the course. |