# Hyperbolic equations

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**Exercise 1**

Consider the problem

$$u_t - u_x = 0, \qquad t \geq 0, \qquad 0 \leq x \leq 1$$

with initial condition

$$u(x, 0) = f(x).$$

and inflow boundary boundary condition

$$u(1, t) = 0$$

Begin by using the method of characteristics to find the exact solution in terms of the initial condition $f(x)$.

**Task 1** Consider the case $f(x) = \sin(2\pi x)$. Implement the Lax-Friedrichs scheme (see the discussion around 7.16 of the notes):

$$U_m^{n+1} = \frac{1}{2}(U_{m+1}^n + U_{m-1}^n) + \frac{k}{2h}(U_{m+1}^n - U_{m-1}^n), \quad m = 1, \ldots, M$$

where $U(x_m, t_n) = U_m^n$. You will need to use a numerical boundary condition at the points $U_0^n$, take

$$U_0^n = -U_1^n + 2U_2^n$$

Using $h = 0.01$ plot the approximate solution $U(x, 0.5)$ at time $t = 0.5$ together with the exact solution. For this purpose, choose a step size $k$ such that $p = -\frac{k}{h}$ takes the values: (i) -1.5, (ii) -0.5, (iii) -0.25. You should find that only (ii) gives a stable method. The instability of (i) is predictable (*why?*) but (iii) is more surprising - indeed the notes prove the stability of this method for all $|p| < 1$! However, the proof concerns only the initial value problem on $-\inf < x < \inf$, and it turns out that the numerical boundary condition at $x = 0$ destroys the stability for $|p| < \frac{1}{3}$. *Hand-in: Plot of the solutions to illustrate stability or lack of it.*

**Task 2** Now take the function

$$f(x) = \begin{cases} 1, & x > 0.5 \\ 0, & x \leq 0.5 \end{cases}$$

Let $h = 0.05$ and plot the approximate solution $U(x, 0.5)$ at time $t = 0.5$ using the Lax-Wendroff method (7.13 of the notes), and the upwind scheme:

$$U_m^{n+1} = U_m^n - \frac{k}{h}(U_{m+1}^n - U_m^n), \quad m = 0, \ldots, M$$

For the Lax-Wendroff method, take the numerical boundary condition

$$U_0^n = U_1^n$$

Take $k$ such that $\frac{k}{h} = \frac{1}{3}$. There are two types of error: a *dissipation* error, where the Fourier modes of the exact solution are damped (i.e. the approximation will be of lower amplitude) and *dispersion* errors, where the Fourier modes are transported at different speeds than the exact solution. Which of the above methods is more dissipative, and which is more dispersive? *Hand-in: Plots of the solution together with a brief comment on the types of error*

**Exercise 2 (Optional)**

Consider the advection diffusion problem

$$u_t + u_x = \nu u_{xx} + f, \quad t \geq 0, \qquad -1 < x < 1, \qquad f \equiv 1$$

with initial condition

$$u(x, 0) = \cos\left(\frac{\pi}{2}x\right)$$

and boundary conditions

$$u(-1, t) = 0, \qquad t \geq 0, \qquad u(1, t) = 0, \qquad t \geq 0.$$

Implement a finite difference discretization for this problem. Use different values of $\nu$, say $\nu = 0.1$, $\nu = 0.01$ $\nu = 0.001$. What do you observe? Experiment with different step size $h$ in space. *Hand-in: Plot the solution for the three different choices of $\nu$ and comment shortly (a couple of sentences) on what you observe.*