NTNU
Norwegian University of
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Department of Mathematical Sciences

Examination paper for
TMA4212 Numerical Solution of Partial Differential Equations By Difference Methods

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Permitted examination support material: C: Approved simple pocket calculator is allowed. The text book by Strikwerda, the book by Süli and Mayers, and the two official notes of the TMA4212 course (98 pages and 28 pages) are allowed. Rottman is allowed. The books in printed version are also allowed. Old exams with solutions are not allowed.

Language: English

Number of pages: 8

Number of pages enclosed: 2

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
sort/hvit ☒	**farger** ☐
skal ha flervalgskjema ☐	

Date Signature

The learning outcomes have been published on the course webpage and in the official description of the course. The seven learning goals **L1** to **L7** are reported in the appendix. Learning outcomes **L3**, **L4** and **L6** have been tested through the project work. Here **L4** is tested further, together with outcomes **L1, L2, L5, L7**.

All answers must be properly argued for.

Problem 1 (**L1, L4, L7**)

Consider the equation

$$u_t = tu_x, \quad 0 < x < 1$$

a) Show that the characteristics of the above equation take the form

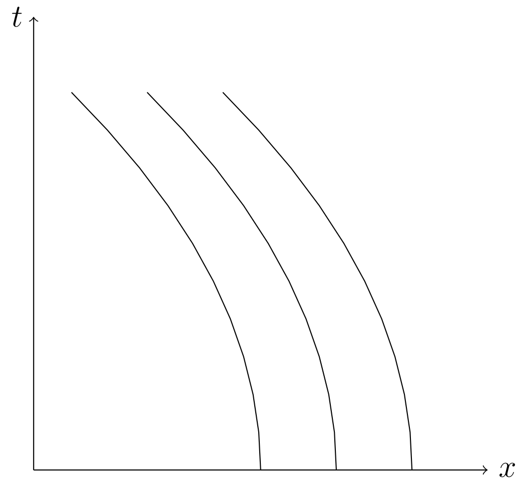
$$x(t) = x_0 - \frac{1}{2}t^2$$

The equation requires a boundary condition to be well-posed. Give an example Dirichlet boundary condition and solve the resulting problem explicitly using the method of characteristics.

Solution: The characteristics $x(t)$ solve the differential equation

$$\frac{dx}{dx} = -t,$$

so that $x(t) = C - \frac{t^2}{2}$. Setting $t = 0$ shows that $C = x(0)$. We see that the inflow



boundary is at $x = 1$, therefore typical boundary conditions would be to give

$$u(x,0) = f(x), \quad u(1,t) = g(t)$$

for some functions f and g. Now to find the solution at an arbitrary point (x,t), we trace back along the characteristic, using that $u(x(t),t)$ is equal for all values of t. To do this, we note that $x_0 = x + \frac{t^2}{2}$, so provided the right hand side is less than one we have

$$u(x,t) = f(x + \frac{t^2}{2})$$

Otherwise, the characteristic exits the domain at the right boundary $x = 1$, and we must find the time of exit τ. The characteristic through (x,t) takes the form

$$X(\tau) = x_0(x,t) - \frac{\tau^2}{2} = x + \frac{t^2 - \tau^2}{2},$$

and we the exit time is such that $X(\tau) = 1$, i.e. $\tau = \sqrt{2x + t^2 - 2}$. To conclude, we find

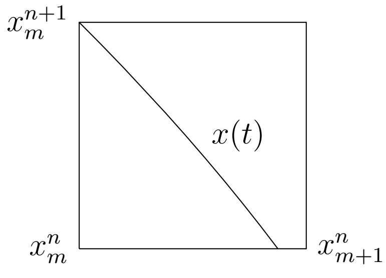
$$u(x,t) = \begin{cases} f(x + \frac{t^2}{2}), & x + \frac{t^2}{2} \leq 1 \\ g(\sqrt{2x + t^2 - 2}), & x + \frac{t^2}{2} > 1 \end{cases}$$

b) Which of the following methods would you use to approximate the solution of the equation?

$$U_m^{n+1} = U_m^n - \frac{nk}{h}(U_{m+1}^n - U_m^n)$$

$$U_m^{n+1} = U_m^n - \frac{nk}{h}(U_m^n - U_{m-1}^n)$$

Solution: We must satisfy the CFL condition to have any hope of convergence. Let us suppose that (x,t) is a nodal point, we sketch the domain of dependence: The characteristic must lie inside the domain of dependence, which is possible if



x_m^{n+1} depends on x_m^n and x_{m+1}^n as per the first method. This is clearly impossible if it is to depend on x_m^n and x_{m-1}^n instead.

c) Suppose you employ your choice of the above methods on a uniform grid with $h = \frac{1}{4}$. What restriction is required on k if the method is to satisfy the CFL condition at all times $t \leq 2$?

Solution: Here we must ensure that $x(t)$ exits each box along the bottom boundary, and not the right hand side. It suffices to check this for the final time step, as it is here the slopes of the characteristics are flattest. The characteristic through the point $(x, 2)$ is of the form $X(t) = x + 2 - \frac{t^2}{2}$, and we require $x \leq X(2-k) \leq x + h$. Using $h = \frac{1}{4}$, we have

$$2 - \frac{(2-k)^2}{2} \leq \frac{1}{4},$$

and hence

$$(2-k)^2 \geq \frac{7}{2}$$

Now as we have $0 \leq k \leq 2$ (otherwise the method would not make sense), we take the positive square root and rearrange to give

$$k \leq 2 - \sqrt{\frac{7}{2}} = 0.1292\ldots$$

Problem 2 (L1, L4)

The diffusion equation

$$u_t = u_{xx}, \quad u(x,0) = f(x)$$

is discretized as follows:

$$U_m^{n+1} - \frac{1}{2}(\frac{k}{h^2} - \frac{1}{6})(U_{m-1}^{n+1} - 2U_m^{n+1} + U_{m+1}^{n+1}) = U_m^n + \frac{1}{2}(\frac{k}{h^2} + \frac{1}{6})(U_{m-1}^n - 2U_m^n + U_{m+1}^n),$$

where $U_m^n = U(mh, nk)$ is the approximate solution on a grid of step sizes h in space and k in time.

a) Find the leading error term of the local truncation error of this method.

Solution: We begin by writing out the Taylor expansions about (mh, nk), for instance:

$$u_{m+1}^{n+1} = u + ku_t + hu_x + \frac{k^2}{2}u_{tt} + \frac{h^2}{2}u_{xx} + khu_{xt} + \frac{k^3}{6}u_{ttt} + \frac{h^3}{6}u_{xxx} + \frac{k^2h}{2}u_{xtt} + \frac{kh^2}{2}u_{xxt} + \ldots$$

where the terms on the right hand side are evaluated at (mh, nk). We then have

$$u_{m-1}^n - 2u_m^n + u_{m+1}^n = h^2u_{xx} + \frac{h^4}{12}u_{xxxx} + \frac{h^6}{360}u_{xxxxxx} + \ldots$$

We now compute $u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}$. Any terms involving only time derivatives (i.e. powers of k alone) will vanish, as they are the same for each of the three terms, and will hence cancel each other out. Similarly, we need only consider terms with even powers of h, resulting in the following:

$$\ldots = \left(h^2 u_{xx} + \frac{h^4}{12} u_{xxxx} + \frac{h^6}{360} u_{xxxxxx}\right) + h^2 k u_{xxt} + \frac{h^4 k}{12} u_{xxxxt} + \ldots$$

Combining the above, we see that the leading order truncation error takes the form

$$\begin{aligned} k\tau_m^n &= k u_t + \frac{k^2}{2} u_{tt} + \frac{k^3}{6} u_{ttt} + \frac{1}{6}(h^2 k u_{xxt} + \frac{h^4 k}{12} u_{xxxxt}) \\ &- \frac{k}{h^2}(h^2 u_{xx} + \frac{h^4}{12} u_{xxxx} + \frac{h^6}{360} u_{xxxxxx} + \frac{h^2 k}{2} u_{xxt} + \frac{h^4 k}{24} u_{xxxxt}) + \ldots \end{aligned}$$

We collect the terms as follows:

$$\begin{aligned} k\tau_m^n &= k(u_t - u_{xx}) + \frac{k^2}{2}(u_{tt} - u_{xxt}) + \frac{kh^2}{12}(u_{xxt} - u_{xxxx}) \\ &+ \frac{k^3}{6} u_{ttt} + \left(\frac{h^4 k}{108} - \frac{h^2 k^2}{24}\right) u_{xxxxt} - \frac{kh^4}{360} u_{xxxxxx} + \ldots \end{aligned}$$

All of the terms on the top line cancel, using $u_t = u_{xx}$, and hence $u_{tt} = u_{xxt}$ etc, so the leading order truncation error is given by the bottom line. These terms can be simplified further if desired, as $u_{xxxxt} = u_{xxxxxx}$.

b) Assume that method is simulated under periodic boundary conditions. Perform a Von Neumann stability analysis.

Solution: Here we substitute $u_m^n = \xi^n e^{i\beta x_m}$, leading to

$$\xi^{n+1}\left(1 - \frac{1}{2}(\frac{k}{h^2} - \frac{1}{6})(e^{i\beta h} - 2 + e^{-i\beta h})\right) = \xi^n\left(1 + \frac{1}{2}(\frac{k}{h^2} + \frac{1}{6})(e^{i\beta h} - 2 + e^{-i\beta h})\right)$$

Now $e^{i\beta h} - 2 + e^{-i\beta h} = -4\sin^2\frac{\beta h}{2}$, from which we conclude

$$\xi = \frac{1 - 2(r - \frac{1}{6})\sin^2\frac{\beta h}{2}}{1 + 2(r + \frac{1}{6})\sin^2\frac{\beta h}{2}},$$

where $r = \frac{k}{h^2}$. The Von Neumann stability condition is $|\xi| \leq 1$ for all $\beta \in \mathbb{R}$. Rewriting the above expression as

$$\xi = \frac{\theta - 2r\sin^2\frac{\beta h}{2}}{\theta + 2r\sin^2\frac{\beta h}{2}},$$

where $\theta = 1 + \frac{1}{3}\sin^2\frac{\beta h}{2}$, we see that the Von Neumann criteria is satisfied for all values of $r > 0$, and hence the method is unconditionally stable.

Problem 3 (**L2, L7**)

We consider the boundary value problem

$$-\frac{d}{dx}\left(\left(x+\frac{1}{3}\right)\frac{du}{dx}\right)+9u=0, \quad 0<x<1, \quad u(0)=1, u(1)=-1.$$

a) Let R be an arbitrary lifting of the boundary conditions, such that $\hat{u}=u-R$ solves a homogeneous Dirichlet problem. Find a variational form of the equation for $\hat{u}$.

Solution: we first multiply by a test function $v \in H_0^1([0,1])$ and integrate, giving

$$\int_0^1 -\frac{d}{dx}\left(\left(x+\frac{1}{3}\right)\frac{du}{dx}\right)v dx+9\int_0^1 uv dx=0,$$

upon integrating by parts we find

$$\int_0^1 (x+\frac{1}{3})u'(x)v'(x)dx+9\int_0^1 u(x)v(x)dx=0$$

We then insert $\hat{u}=u-R$, obtaining the equation

$$\int_0^1 (x+\frac{1}{3})\hat{u}'(x)v'(x)dx+9\int_0^1 \hat{u}(x)v(x)dx=-\int_0^1 (x+\frac{1}{3})R'(x)v'(x)dx-9\int_0^1 R(x)v(x)dx.$$

The variational form of the equation for $\hat{u}$ is to find $\hat{u} \in H_0^1([0,1])$ such that the above identity holds for all $v \in H_0^1([0,1])$.

b) Suppose we approximate the solution of the given equation by the finite element method on a uniform grid

$$0=x_0<x_1<\ldots<x_M<x_{M+1}=1,$$

using linear nodal basis functions. Show that this results in a linear system

$$A\hat{u}=b,$$

and argue that A is positive definite, and hence that there exists a unique solution to the discrete problem (you should not calculate the entries of A or b explicitly at this stage, simply indicate how they can be found).

Solution: we expand $\hat{u}=\sum_i u_i\varphi_i$ and $v=\sum_j v_j\varphi_j$ in terms of the basis functions φ_i, obtaining

$$\sum_{i,j} u_i v_j \left(\int_0^1 (x+\frac{1}{3})\varphi_i'\varphi_j'+9\int_0^1 \varphi_i\varphi_j\right)=\sum_j v_j\left(-\int_0^1 R'\varphi_j-9\int_0^1 R\varphi_j\right),$$

or in matrix/vector form

$$v^T(A+9M)u = v^T b,$$

where

$$A_{ij} = \int_0^1 (x+\frac{1}{3})\varphi_i'\varphi_j', \quad M_{ij} = \int_0^1 \varphi_i\varphi_j, \quad b_j = -\int_0^1 R'\varphi_j - 9\int_0^1 R\varphi_j$$

As this must hold for all v, we obtain the equation $(A+9M)u = b$, where A, M and b are as above. Now A and M are clearly symmetric, hence so is $A+9M$. To obtain positive definiteness, we require

$$v^T(A+9M)v > 0 \quad \forall v \neq 0.$$

One possible proof is to relate this expression to the norm in H_0^1, as follows

$$v^T(A+9M)v = v_i^2\int_0^1 \left(9\varphi_i^2 + (x+\frac{1}{3})\varphi_i'^2\right) \geq \frac{1}{3}||v||_{H_0^1} > 0$$

Any linear equation $Au = b$ where A is positive definite must have a unique solution (as, e.g. all the eigenvalues of A are strictly positive and hence non-zero, so A is non-singular), hence we are done.

c) Suppose $M = 2$, such that the grid is $x_0 = 0, x_1 = \frac{1}{3}, x_2 = \frac{2}{3}, x_3 = 1$. Find the resulting approximate solution u.

Solution: we begin by finding A, M and b, then implement boundary conditions and solve the system. As usual, we construct the matrices/vectors elementwise. First, to construct A, the contribution coming from the element $[x_i, x_{i+1}]$ is

$$\int_{x_i}^{x_{i+1}} \frac{(x+\frac{1}{3})}{h^2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} dx = 9\left((x_{i+1}-x_i)^2 + \frac{1}{3}\right)\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$$

The three elements $[0,\frac{1}{3}]$, $[\frac{1}{3},\frac{2}{3}]$ and $[\frac{2}{3},1]$ give respectively the submatrices

$$\begin{pmatrix}\frac{3}{2} & \frac{3}{2} & 0 & 0\\ -\frac{3}{2} & -\frac{3}{2} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & \frac{5}{2} & -\frac{5}{2} & 0\\ 0 & -\frac{5}{2} & \frac{5}{2} & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \frac{7}{2} & -\frac{7}{2}\\ 0 & 0 & -\frac{7}{2} & \frac{7}{2}\end{pmatrix},$$

upon adding together we obtain

$$A = \begin{pmatrix}\frac{3}{2} & -\frac{3}{2} & 0 & 0\\ -\frac{3}{2} & 4 & -\frac{5}{2} & 0\\ 0 & -\frac{5}{2} & 6 & -\frac{7}{2}\\ 0 & 0 & -\frac{7}{2} & \frac{7}{2}\end{pmatrix}$$

The matrix M is the standard mass matrix, the contributions from each element are the same and can be shown to equal

$$\frac{h}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Indeed, it is easiest to show this on the element $[0, h]$, where $\varphi_0 = \frac{x}{h}$ and $\varphi_1 = 1 - \frac{x}{h}$. The off-diagonal terms are given by

$$\frac{1}{h^2}\int_0^h x(h-x)dx = \frac{1}{h^2}\int_0^h (xh - x^2)dx = \frac{1}{h^2}(\frac{h^3}{2} - \frac{h^3}{3}) = \frac{h}{6},$$

whilst by symmetry considerations the diagonal terms must be the same. We then compute

$$\int_0^h \varphi_0^2 = \frac{1}{h^2}\int_0^h x^2 dx = \frac{1}{h^2}\frac{h^3}{3} = \frac{h}{3}.$$

As $9\frac{h}{6} = \frac{1}{2}$, we have

$$9M = \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 2 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 2 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 1 \end{pmatrix}$$

Combining the above leads to

$$A + 9M = \begin{pmatrix} \frac{5}{2} & -1 & 0 & 0 \\ -1 & 6 & -2 & 0 \\ 0 & -2 & 8 & -3 \\ 0 & 0 & -3 & \frac{9}{2} \end{pmatrix}$$

The vector b takes the form $-(A + 9M) \cdot (1, 0, 0, -1)^T$, i.e. we have

$$b = -\begin{pmatrix} \frac{5}{2} & -1 & 0 & 0 \\ -1 & 6 & -2 & 0 \\ 0 & -2 & 8 & -3 \\ 0 & 0 & -3 & \frac{9}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{5}{2} \\ 1 \\ -3 \\ \frac{9}{2} \end{pmatrix}$$

We then implement boundary conditions and solve for $\hat{u}$:

$$\begin{pmatrix} 6 & -2 \\ -2 & 8 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

this gives

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{22}\begin{pmatrix} 1 \\ -8 \end{pmatrix}$$

Adding in the boundary values according to the lifting leaves us with

$$\begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{22} \\ -\frac{4}{11} \\ -1 \end{pmatrix}$$

Problem 4 (**L5**)

Let $\alpha, \beta \in \mathbb{R}$, and consider the linear system

$$\begin{pmatrix} \alpha & \beta & 0 \\ \beta & \alpha & \beta \\ 0 & \beta & \alpha \end{pmatrix} u = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

a) Perform two steps of the Jacobi iteration starting from $u_0 = (0, 0, 0)^T$.

Solution: The Jacobi iteration takes the form

$$u_{k+1} = -\begin{pmatrix} 0 & \frac{\beta}{\alpha} & 0 \\ \frac{\beta}{\alpha} & 0 & \frac{\beta}{\alpha} \\ 0 & \frac{\beta}{\alpha} & 0 \end{pmatrix} u_k + \begin{pmatrix} \frac{1}{\alpha} \\ 0 \\ -\frac{1}{\alpha} \end{pmatrix}$$

If we start from $u^0 = 0$, we have $u^1 = \frac{1}{\alpha}(1, 0, -1)^T$, and hence

$$u_2 = \frac{1}{\alpha^2} \begin{pmatrix} 0 & \beta & 0 \\ \beta & 0 & \beta \\ 0 & \beta & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} \frac{1}{\alpha} \\ 0 \\ -\frac{1}{\alpha} \end{pmatrix} = \begin{pmatrix} \frac{1}{\alpha} \\ 0 \\ -\frac{1}{\alpha} \end{pmatrix}$$

b) For what values of α, β does the Jacobi iteration converge?

Solution: Here the question is a little unclear - with the given starting value the solution is found after one iteration, so we have convergence as long as $\alpha \neq 0$. More generally, the eigenvalues of the iteration matrix are given solving the characteristic equation

$$\lambda^3 - 2\lambda(\frac{\beta}{\alpha})^2 = 0,$$

which has solution $\lambda = 0, \pm\sqrt{2}\frac{\beta}{\alpha}$. The condition $\rho(A) < 1$ would lead to $\sqrt{2}\left|\frac{\beta}{\alpha}\right| < 1$.

Appendix

- **Jacobi iteration**: given the linear system of equations

$$Ax = b$$

with A $n \times n$ matrix and b a vector with n components, we split A as the sum of its diagonal D minus a matrix R:

$$A = D - R.$$

We assume that D is invertible.

The Jacobi iteration is an iterative method to approximate the solution of the linear system, and is given by the iteration

$$x^{k+1} = D^{-1}(R\,x^k + b), \tag{1}$$

with x^0 a given initial guess. Note that (1) this is a fixed point iteration to solve the fixed point equation $x = D^{-1}(R\,x + b)$, whose solution is the same as for the linear system.

Learning outcome:

Knowledge	**L1**	Understanding of error analysis of difference methods: consistency, stability, convergence of difference schemes.
	L2	Understanding of the basics of the finite element method.
Skills	**L3**	Ability to choose and implement a suitable discretization scheme given a particular PDE, and to design numerical tests in order to verify the correctness of the code and the order of the method.
	L4	Ability to analyze the chosen discretization scheme, at least for simple PDE-test problems.
	L5	Ability to attack the numerical linear algebra challenges arising in the numerical solution of PDEs.
General competence	**L6**	Ability to present in oral and written form the numerical and analytical results obtained in the project work.
	L7	Ability to apply acquired mathematical knowledge in linear algebra and calculus to achieve the other goals of the course.