Norwegian University of
Science and Technology
Department of Mathematical Sciences

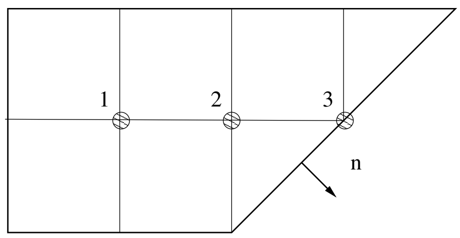
Solution TMA4212

07. juni 2010

Problem 1 We consider the Laplace equation

$$\Delta u = 0, \quad \text{on } \Omega. \tag{1}$$

The domain Ω and the grid are given in the picture and the mesh size is $h = 0.25$.



We want to find the numerical approximation $U_i \approx u(x_i, h)$, $i = 1, 2, 3$ and $x_i = i\,h$. Using the five-point formula and the boundary conditions,

$$\begin{aligned} u(x,0) &= 0, && 0 \le x \le 0.5, \\ u(x,0.5) &= \sin \pi x, && 0 \le x \le 1, \\ u(0,y) &= 0, && 0 \le y \le 0.5, \end{aligned}$$

we obtain:

$$\begin{aligned} \sin \pi h + U_2 - 4U_1 &= 0, \\ U_1 + \sin \pi 2h + U_3 - 4U_2 &= 0. \end{aligned}$$

To obtain the last equation of the linear system, we derive an approximation of the Neumann boundary condition along $0 \le y \le 0.5$, $x = y + 0.5$. This condition is

$$0 = \left.\frac{\partial u}{\partial n}\right|_{(x,y)} = n_x u_x + n_y u_y$$

where n_x and n_y are the components of the normal vector (of length 1) pointing outside the domain. It is easily seen that $n_x = \frac{1}{\sqrt{2}}$ and $n_y = -\frac{1}{\sqrt{2}}$ and

$$\left.\frac{\partial u}{\partial n}\right|_{(x,y)} = \frac{1}{\sqrt{2}}(u_x - u_y).$$

We now replace derivatives with difference approximations and obtain

$$\left.\frac{\partial u}{\partial n}\right|_{(x_3,h)} = \frac{1}{\sqrt{2}}\left[\left(\frac{u(x_3,h)-u(x_2,h)}{h} - \frac{\sin\pi x_3 - u(x_3,h)}{h}\right) + \frac{h}{2}(u_{xx}(\xi,h)+u_{yy}(\eta,h))\right],$$

where $\xi \in (x_2, x_3),\ \eta \in (h, 2h)$, and we have used backward differences in the x direction and forward differences in the y direction. The truncation error is $\mathcal{O}(h)$. Since the five-point formula is of second order in h the overall method is consistent and has a truncation error $\mathcal{O}(h)$.

By replacing $u(x_i, h)$ with U_i in the previous formula we get

$$\left.\frac{\partial u}{\partial n}\right|_{(x,y)} \approx \frac{1}{\sqrt{2}}\left(\frac{U_3-U_2}{h} - \frac{\sin\pi 3h - U_3}{h}\right),$$

setting the right hand side of the last expression equal to zero we obtain the third equation of the linear system. After rearranging the terms we get

$$\begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = -\begin{bmatrix} \sin\pi h \\ \sin\pi 2h \\ \sin\pi 3h \end{bmatrix}.$$

Problem 2

a) We want to apply the finite element method to

$$-u_{xx} = f, \quad x \in [0,1], \quad u(0) = 1,\ u(1) = 0,\ f(x) = -\frac{\pi^2}{4}\cos\left(x\,\frac{\pi}{2}\right), \tag{2}$$

to this end we derive the Galerkin formulation of the problem. We start by multiplying both sides of the equation by a test function chosen arbitrarily in H_0^1 and we obtain

$$-\int_0^1 u_{xx} v\,dx = \int_0^1 f v\,dx.$$

Integrating by parts and since v is zero on the boundary this gives

$$\int_0^1 u_x v_x\,dx = \int_0^1 f v\,dx.$$

The Galerkin formulation of the problem is: find $u \in H_E^1$ such that

$$a(u,v) = \langle f, v\rangle, \quad \forall v \in H_0^1,$$

and we have used the notation $\langle\cdot,\cdot\rangle$ to denote the L^2 inner product. Solutions of (2) are also solutions of the Galerkin formulation, but not necessarily the other way around.

b) The numerical approximation given by the Galerkin method is the solution of the problem: find $u \in V_E^1$ such that

$$a(u, v) = \langle f, v\rangle, \quad \forall v \in V_0^1,$$

where $V_0^1 = \{v \in H_0^1 \,|\, v = \sum_{j=1}^{M-1} \phi_j v_j\}$ and $V_E^1 = \{v \in H_E^1 \,|\, v = \phi_0 + \sum_{i=1}^{M-1} \phi_i v_i\}$. By taking $v = \phi_j$, $j = 1, \dots, M-1$, and expressing u in the Galerkin method by means of the basis functions, i.e.

$$u = \phi_0 + \sum_{i=1}^{M-1} u_i \phi_i, \quad U = [u_1, \dots, u_{M-1}]^T,$$

we obtain the requested linear system of equations:

$$\sum_{i=1}^{M-1} u_i a(\phi_i, \phi_j) = \langle f, \phi_j\rangle - a(\phi_0, \phi_j), \quad j = 1, \dots, M-1.$$

The matrix C of the linear system is an $(M-1) \times (M-1)$-matrix with entries

$$C_{j,i} := a(\phi_i, \phi_j) = \int_0^1 \phi_i' \phi_j' \, dx, \quad i, j = 1, \dots, M-1,$$

with $\phi_i' = \frac{d\phi_i}{dx}$. By using the expressions for the basis functions and computing the integrals we obtain:

$$\begin{aligned} C_{i,j} &= C_{j,i} \\ C_{i-1,i} = C_{i,i+1} &= \int_{x_{i-1}}^{x_i} \phi_i' \, \phi_{i-1}' \, dx = -\frac{1}{h}, \\ C_{i,i} &= \int_{x_{i-1}}^{x_{i+1}} (\phi_i')^2 \, dx = \frac{2}{h}. \end{aligned}$$

c) We get

$$b_j = \langle f, \phi_j\rangle - a(\phi_0, \phi_j), \quad j = 1, \dots, M-1,$$

and we note that $a(\phi_0, \phi_j) = 0$ for $j = 2, \dots, M-1$. So we have

$$b_j = -\frac{\pi^2}{4h} \int_{x_{j-1}}^{x_j} \cos(x\frac{\pi}{2})(x - x_{j-1}) \, dx - \frac{\pi^2}{4h} \int_{x_j}^{x_{j+1}} \cos(x\frac{\pi}{2})(x_{j+1} - x) \, dx, \quad j = 2, \dots, M-1$$

and using the trapezoidal rule to approximate the integrals we get

$$b_j \approx -\frac{\pi^2 \, h}{4} \cos(x_j \frac{\pi}{2}) \quad j = 2, \dots, M-1,$$

and

$$b_1 \approx -\frac{\pi^2 \, h}{4} \cos(x_1 \frac{\pi}{2}) + \frac{1}{h}.$$

Problem 3

a) We consider the equation

$$u_t = -u_{xx} - u_{xxxx}, \quad u(0) = u(1) = 0, \quad x \in [0,1].$$

We consider the grid $x_m = hm$, $h = 1/M$, $m = 0, \ldots, M$. Discretizing by central differences and the trapezoidal rule in time we get

$$U_m^{n+1} = U_m^n + \frac{k}{2h^2}\big(- \delta_x^2 (U_m^n + U_m^{n+1}) - \frac{1}{h^2}\delta_x^4 (U_m^n + U_m^{n+1})\big),$$

where as usual

$$\delta_x^2 U_m^n = U_{m+1}^n - 2U_m^n + U_{m-1}^n, \quad m = 1, \ldots, M-1,$$

and by straightforward calculation

$$\delta_x^4 U_m^n = U_{m+2}^n - 4U_{m+1}^n + 6U_m^n - 4U_{m-1}^n + U_{m-2}^n, \quad m = 2, \ldots, M-2$$

corresponding to the entries of the rows of B^2 for $m = 2, \ldots, M-2$. Since $U_0^n = 0$ and $U_M^n = 0$ we have

$$\delta_x^4 U_1^n = \delta_x^2 (U_2^n - 2U_1^n) = U_3^n - 4U_2^n + 5U_1^n,$$

(corresponding to the first component of $B^2 U^n$),

$$\delta_x^4 U_{M-1}^n = \delta_x^2 (U_{M-2}^n - 2U_{M-1}^n) = U_{M-3}^n - 4U_{M-2}^n + 5U_{M-1}^n,$$

(corresponding to the last component of $B^2 U^n$). In matrix format the method can be expressed by

$$U^{n+1} = U^n + r\,(-B - \frac{1}{h^2}B^2)(U^n + U^{n+1}), \quad r := \frac{k}{2h^2}$$

where B is the usual discrertization of the Laplace operator. The method can then be written in the form

$$AU^{n+1} = DU^n,$$

where

$$A = I + r\,(B + \frac{1}{h^2}B^2), \quad D = I - r\,(B + \frac{1}{h^2}B^2).$$

Now A and D are symmetric and therefore diagonalizable via an orthogonal transformation and have the same eigenvectors as B. To discuss the invertibility of A we consider its eigenvalues which are

$$\lambda_m^A = 1 + r\,(\lambda_m^B + \frac{1}{h^2}(\lambda_m^B)^2), \quad \lambda_m^B = -\gamma^2, \quad \gamma = 2\sin(\frac{m\,\pi h}{2}),$$

where $0 \leq \sin(\frac{m\,\pi\,h}{2}) \leq 1$ for $m = 1, \ldots, M-1$ and $h = 1/M$, and

$$\lambda_m^A = 1 + r\,\gamma^2 (\frac{\gamma^2}{h^2} - 1),$$

$$\lambda_m^A = 1 + r\,\gamma^2(\frac{\gamma}{h} - 1)(\frac{\gamma}{h} + 1).$$

The term $(\frac{\gamma}{h} - 1) = (\frac{2}{h}\sin(\frac{m\,\pi\,h}{2}) - 1)$ is positive for all $h = 1/M$ and $M \geq 2$, all the other terms in the above expression are positive so λ_m^A is different from zero for $m = 1, \dots, M-1$ and A is invertible. Thus we choose $H = 1$.

b) We assume $M \geq 1$ and $h = 1/M \leq 1$ and write the method in the form

$$U^{n+1} = CU^n, \quad C = A^{-1}D.$$

Since C is a symmetric matrix, to show Lax-Richtmyer stability for the method it is sufficient to show that the spectral radius of C, $\rho(C)$, is less than or equal to 1. The eigenvalues of C are

$$\lambda_m^C = \frac{1 - r\,\gamma^2(\frac{\gamma^2}{h^2} - 1)}{1 + r\,\gamma^2(\frac{\gamma^2}{h^2} - 1)},$$

as in the previous question we see that $\gamma^2(\frac{\gamma^2}{h^2} - 1)$ is positive for all $h \leq 1$ and $m = 1, \dots, M-1$; so we get that $|\lambda_m^C| \leq 1$ for $m = 1, \dots, M-1$, and $\rho(C) \leq 1$.

c) We consider the method componentwise

$$U_m^{n+1} = U_m^n + r\left(-\delta_x^2 - \frac{\delta_x^4}{h^2}\right)(U_m^n + U_m^{n+1}).$$

We assume

$$U_m^n = \xi_\beta^n\, e^{\mathrm{i}\beta x_m}, \quad x_m = mh$$

and substitute in the previous equation. After some algebra we get

$$\xi_\beta = \frac{1 - r\gamma^2(\frac{\gamma^2}{h^2} - 1)}{1 + r\gamma^2(\frac{\gamma^2}{h^2} - 1)}, \quad \gamma^2 = -(e^{\mathrm{i}\beta h} - 2 + e^{-\mathrm{i}\beta h}) = 4\sin^2(\frac{\beta h}{2}).$$

The rest of the analysis consists in proving that $|\xi_\beta| \leq 1$ and it is the same as in the previous question.