Hyperbolic equations

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Exercise 1

Consider the advection diffusion problem

$$u_t + u_x = \nu u_{xx} + f, \quad t \geq 0, \qquad -1 < x < 1, \qquad f \equiv 1$$

with initial condition

$$u(x, 0) = \cos\left(\frac{\pi}{2} x\right)$$

and boundary conditions

$$u(-1, t) = 0, \qquad t \geq 0, \qquad u(1, t) = 0, \qquad t \geq 0.$$

Implement a finite difference discretization for this problem. Use different values of ν, say $\nu = 0.1$, $\nu = 0.01$ $\nu = 0.001$. What do you observe? Experiment with different step size h in space. *Hand-in: Plot the solution for the three different choices of* ν *and comment shortly (a couple of sentences) on what you observe.*

Exercise 2

Consider the problem

$$u_t + a u_x = 0, \qquad t \geq 0, \qquad -\infty < x < \infty$$

with initial condition

$$u(x, 0) = g(x).$$

Task 1 Implement the explicit difference formulae of chapter 7.3 in the note, in particular (7.11) and (7.12), and verify that (7.12) is always unstable while (7.11) converges. *Hand-in: Plot of the solutions to illustrate stability or lack of it.*
How to choose g? You can't represent the solution on the whole real line. You may either use a periodic function or choose a function with compact support.

Task 2 Implement Lax-Wendroff and Leap-Frog and verify numerically their order in space and time. *Hand-in: Plot of the solutions and convergence plots that verify correct order in space and time.*
Hint: You need to choose g smooth enough to verify the correct order (why?).

Task 3 (Optional) Consider finally the inviscid Burgers equation

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0.$$

Discretize the time derivative and the space derivative in this equation using the same approximation as in formula (7.11). Verify numerically that the method converges. *Hand-in: Plot of the solution and convergence plots that verify correct order in space and time.*