NTNU – Trondheim
Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for

TMA4212 Numerical solution of differential equations with difference methods

Academic contact during examination: Elena Celledoni

Phone: 48238584

Examination date: 28. May 2016

Examination time (from–to): 09:00-13:00

Permitted examination support material: C: Approved simple pocket calculator is allowed. Rottman is allowed.

Language: English

Number of pages: 3

Number of pages enclosed: 2

Checked by:

Date Signature

The learning outcome has been published on the course webpage and on the official description of the course. The seven learning goals **L1** to **L7** are reported in the appendix. Learning outcome **L6**, **L3** and to some extent **L4** have been tested through the project work. We here test further the achievement of **L4** as well as **L1, L2, L5 L7**. All answers must be properly argued for.

Problem 1 (L2)

We are solving the Poisson equation

$$\Delta u = u_{xx} + u_{yy} = f, \qquad \text{in}\,\Omega, \qquad u = 0 \quad \text{on}\,\partial\Omega,$$

with the finite element method. $\Omega \subset \mathbf{R}^2$ is a rectangular domain with the sides aligned with the x and y axes and one corner in the origin. We use square elements and quadratic basis functions. We consider the element K with vertices $(0,0)$, $(h,0)$, $(0,h)$ and (h,h), where h is a discretization parameter.

a) Find an expression for the four quadratic, finite element basis functions $\varphi_1 = \varphi_{(0,0)}$, $\varphi_2 = \varphi_{(h,0)}$, $\varphi_3 = \varphi_{(0,h)}$ and $\varphi_4 = \varphi_{(h,h)}$ on K, by combining appropriately the linear polynomials

$$\frac{h-x}{h}, \quad \frac{x}{h}, \quad \frac{h-y}{h}, \quad \frac{y}{h}.$$

b) Find the bilinear function α arising in the Galerkin formulation of the Poisson equation. The element stiffness matrix is

$$A^K = \{\alpha_K(\varphi_i, \varphi_j)\}_{i,j=1,\dots,4},$$

where α_K denotes the restriction of α to the element K. Find the elements $A^K_{2,4}$ and $A^K_{4,2}$ of this matrix.

Problem 2 (L1, L3)

Consider the linear advection equation

$$u_t + au_x = 0, \quad x \in \mathbf{R}, \quad u(x,0) = u_0(x)$$

with a constant. Consider the two schemes

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} + a\frac{u_{m+1}^n - u_m^n}{\Delta x} = 0, \tag{1}$$

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} + a\frac{u_m^n - u_{m-1}^n}{\Delta x} = 0. \tag{2}$$

a) Which one of the two schemes would you use to approximate this equation when $a > 0$ and which one when $a < 0$ and why?

b) Perform a von Neumann stability analysis for the scheme (1).

Problem 3 (**L5, L7**)

a) The 2×2 matrix A is symmetric and positive definite. Show that the Jacobi iteration for $Ax = b$ converges. For the Jacobi iteration see the appendix.

b) The $N \times N$ matrix E has all its elements equal to 1. Show that one of the eigenvalues of E is N, and all the others are zero. Construct a matrix $A = I + \kappa E$, where κ is a constant to be determined, such that A is symmetric and positive definite, but in general the Jacobi method diverges.

c) Explain why the result of **b)** does not contradict point **a)** above in this exercise.

Problem 4 (**L1, L2, L4, L7**)
Consider the boundary value problem

$$-p_0 u'' + r_0 u = f(x), \quad u(0) = 0, \quad u(1) = 0, \tag{3}$$

on the interval $[0, 1]$, where p_0 and r_0 are positive constants and $f \in C^4[0, 1]$. Use equally spaced points

$$x_i = ih, \quad i = 0, 1, \ldots, n, \quad \text{with } h = \frac{1}{n},\ n \geq 2,$$

and the standard piecewise linear finite element basis functions (hat functions) φ_i, $i = 1, 2, \ldots, n-1$.

a) State the weak formulation of the problem and the Galerkin method and show that the finite element equations for $u_i = u^h(x_i)$ become

$$-p_0 \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + r_0 \frac{u_{i-1} + 4u_i + u_{i+1}}{6} = \frac{1}{h}\langle f, \varphi_i \rangle \tag{4}$$

for $i = 1, 2, \ldots, n-1$, with $u_0 = 0$ and $u_n = 0$.

b) By expanding in Taylor series we have obtained that

$$\frac{1}{h}\langle f, \varphi_i \rangle = f(x_i) + \frac{1}{12}h^2 f''(x_i) + \mathcal{O}(h^4). \tag{5}$$

Interpreting (4) as a finite difference approximation to the boundary value problem, and using (5), show that the corresponding local truncation error τ_i satisfies

$$\tau_i = \frac{1}{12}h^2 r_0 u''(x_i) + \mathcal{O}(h^4), \quad i = 1, \ldots, n-1.$$

c) Show finally the following bound for the error

$$\max_{0 \le i \le n} |u(x_i) - u^h(x_i)| \le M h^2,$$

where M is a positive constant.

Appendix

- $$\frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2} = u''(x_i) + \frac{h^2}{12}u''''(x_i) + \mathcal{O}(h^4)$$

- **Jacobi iteration**: given the linear system of equations

  $$Ax = b$$

  with A $n \times n$ matrix and b a vector with n components, we split A as the sum of its diagonal D minus a matrix R:

  $$A = D - R.$$

  We assume that D is invertible.

  The Jacobi iteration is an iterative method to approximate the solution of the linear system, and is given by the iteration

  $$x^{k+1} = D^{-1}(R\,x^k + b), \tag{6}$$

  with x^0 a given initial guess. Note that (6) this is a fixed point iteration to solve the fixed point equation $x = D^{-1}(R\,x + b)$, whose solution is the same as for the linear system.

Learning outcome:

Knowledge	**L1**	Understanding of error analysis of difference methods: consistency, stability, convergence of difference schemes.
	L2	Understanding of the basics of the finite element method.
Skills	**L3**	Ability to choose and implement a suitable discretization scheme given a particular PDE, and to design numerical tests in order to verify the correctness of the code and the order of the method.
	L4	Ability to analyze the chosen discretization scheme, at least for simple PDE-test problems.
	L5	Ability to attack the numerical linear algebra challenges arising in the numerical solution of PDEs.
General competence	**L6**	Ability to present in oral and written form the numerical and analytical results obtained in the project work.
	L7	Ability to apply acquired mathematical knowledge in linear algebra and calculus to achieve the other goals of the course.