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Department of Mathematical Sciences

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EXAM IN TMA4212

June 7th 2010

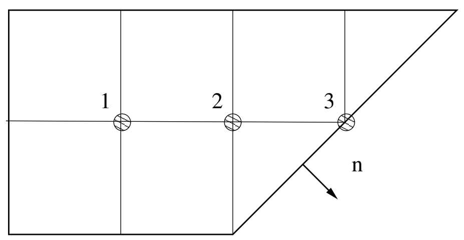
Time: 09:00–13:00

Allowed material: B – All printed and handwritten material is allowed.
A simple calculator is allowed.
All exam questions are given equal weight.

Problem 1 Consider the Laplace equation

$$\Delta u = 0, \quad \text{on } \Omega. \tag{1}$$

The domain Ω is depicted in the figure



with the origin in the lower left corner. Consider the following boundary conditions:

$$\begin{aligned} u(x,0) &= 0, & 0 \le x \le 0.5, \\ u(x,0.5) &= \sin \pi x, & 0 \le x \le 1, \\ u(0,y) &= 0, & 0 \le y \le 0.5, \\ \left.\frac{\partial u}{\partial n}\right|_{(x,y)} &= 0 & 0 \le y \le 0.5,\ x = y + 0.5, \end{aligned}$$

where $\frac{\partial u}{\partial n} = n_x u_x + n_y u_y$, and n_x, n_y are the components of the normal vector (with Euclidean length 1) pointing outside the domain Ω (as shown in the picture). Consider the grid in the figure with step-size $h = 0.25$ both in the

x- and the y-direction. Use the 5-point formula and a consistent approximation of the boundary conditions on the right boundary. Show that the method is consistent. Find the 3×3 system of linear equations whose solution is the numerical approximation in the nodes denoted with the indexes $1, 2, 3$.

Problem 2 We are looking for a discretization of the following equation by the finite element method,

$$-u_{xx} = f, \quad x \in [0, 1], \quad u(0) = 1,\ u(1) = 0,\ f(x) = -\frac{\pi^2}{4}\cos(x\,\frac{\pi}{2}), \qquad (2)$$

using piecewise linear basis functions on the grid $x_0, \ldots, x_M$ with $x_m = mh$ and $h = 1/M$.

a) Use the bilinear form

$$a(u, v) = \int_0^1 u_x v_x \, dx,$$

to write the Galerkin formulation of the problem using appropriate function spaces[1], assume $u \in H^1_E$ and $v \in H^1_0$. Explain the connection between the Galerkin formulation and (2).

b) Find the $(M-1) \times (M-1)$-matrix C such that

$$CU = b,$$

is the linear system corresponding to the Galerkin method. To find the elements of C, compute the integrals exactly.

c) Find b. Use the trapezoidal rule to approximate the integrals.

Problem 3 Consider the equation

$$u_t = -u_{xx} - u_{xxxx}, \quad u(0) = u(1) = 0, \quad x \in [0, 1].$$

Consider the grid $x_m = hm$, $h = 1/M$, $m = 0, \ldots, M$. We will discretize the problem with central differences in space and with the trapezoidal integration method in time (the Crank-Nicolson method). Let $u(x_0, t) = u(x_M, t) = 0$ and let k be the step-size.

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[1]

$$\begin{aligned}
H^1((0,1)) &:= \{v \in L^2((0,1)) \,|\, v \text{ absolutely continuous on } [0,1],\ \partial_x v \in L^2((0,1))\,\},\\
H^1_0((0,1)) &:= \{v \in H^1((0,1)) \,|\, v(0) = v(1) = 0\,\},\\
H^1_E((0,1)) &:= \{v \in H^1((0,1)) \,|\, v(0) = 1,\ v(1) = 0\,\}.
\end{aligned}$$

a) Use the following approximation of the fourth derivative:

$$u_{xxxx}|_{x_m} = \frac{\delta_x^4 u(x_m)}{h^4} + \mathcal{O}(h^2),$$

where

$$\delta_x^2 u(x_m) := u(x_{m+1}) - 2u(x_m) + u(x_{m-1}), \quad \delta_x^4 u(x_m) = \delta_x^2\, \delta_x^2 u(x_m).$$

Let $\frac{1}{h^2}B$ be the discrete Laplace operator given in the formulae in the last page of this document, a $(M-1)\times(M-1)$-matrix, and let $U^n := [U_1^n, \dots, U_{M-1}^n]^T$ be the numerical approximation.

Consider the components of $\frac{1}{h^4}B^2U^n$. Explain how they can be interpreted as approximations of $u_{xxxx}|_{(x_m,t_n)}$; look separately at the cases $m = 1$, $m = 2, \dots, M-2$, and $m = M-1$.

Show that the Crank-Nicolson method can be written in the form

$$AU^{n+1} = DU^n,$$

and find A and D as functions of B.

Show that there exists a constant H such that A^{-1} exists for all $h < H$.

b) Write the method in the form

$$U^{n+1} = CU^n, \quad C = A^{-1}D,$$

and show that the method is Lax-Richtmyer stable. Assume $M \geq 2$ such that $h = 1/M \leq 1$.

c) Consider now periodic boundary conditions $u(x) = u(x+1)$, such that it is possible to perform a von Neumann stability analysis. Show that the method is von Neumann stable.

Piecewise linear finite element basis functions

$$\phi_j(x) = \begin{cases} \frac{(x-x_{j-1})}{h}, & x_{j-1} \le x \le x_j, \\ \frac{(x_{j+1}-x)}{h}, & x_j \le x \le x_{j+1}, \\ 0, & \text{otherwise}, \end{cases} \qquad j = 1, \dots, M-1,$$

$$\phi_M(x) = \begin{cases} \frac{(x-x_{M-1})}{h}, & x_{M-1} \le x \le x_M, \\ 0, & \text{otherwise}, \end{cases} \qquad \phi_0(x) = \begin{cases} \frac{(x_0-x)}{h}, & x_0 \le x \le x_1, \\ 0, & \text{otherwise}. \end{cases}$$

Eigenvalues of the discrete Laplace operator

Consider the $(M-1) \times (M-1)$ matrix $B = \text{tridiag}(1, -2, 1)$, $\frac{1}{h^2}B$, with $h = \frac{1}{M}$, obtained by discretizing the Laplace operator with homogeneous Dirichlet boundary conditions. The eigenvalues of B are given as

$$\lambda_m(B) = 2\left(\cos(m\pi h) - 1\right) = -4\sin^2\left(\frac{m\pi h}{2}\right), \quad m = 1, \dots, M-1.$$

Trapezoidal rule for numerical quadrature

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}(f(a) + f(b)).$$

Lax-Richtmyer stability is discussed in chapter 4.6 of the notes and in definition 9.1 in the book *Finite difference methods for ordinary and partial differential equations* by Randall J. LeVeque.