Solution to the exam SIF5045 Numeriske Differentialligninger, May 2000

Problem 1

a) According to the Lemma 3.5 from the book a collocation method with collocation points c_1 and c_2 is identical to the Runge–Kutta method with coefficients

$$a_{j,i} = \int_0^{c_j} \frac{q_i(t)}{q_i(c_i)} dt, \qquad i, j = 1, 2;$$

$$b_j = \int_0^1 \frac{q_j(t)}{q_j(c_j)} dt, \qquad j = 1, 2;$$

where $q_1 = (t - c_2)$ and $q_2 = (t - c_1)$. If we require that the coefficients of the tableau are generated according to these formulas, we find the unique solution: the collocation method with collocation points $c_1 = (2 - \sqrt{2})\mu$ and $c_2 = (2 + \sqrt{2})\mu$

b) It is easy to check that the second order conditions

$$\sum_i b_i = b_1 + b_2 = 1, \qquad \sum_i b_i c_i = b_1 c_1 + b_2 c_2 = \frac{1}{2}$$

are satisfied for any $\mu \in \mathbf{R}$.

Remark We use order conditions for RK methods. Alternatively we can use Theorem 3.7 from the book (Arieh Iserles: Numerical Analysis of Differential Equations) to check the order of the collocation method.

c) The method has the third order if in addition to the second order conditions (point **b)**) two additional conditions

$$\sum_i b_i c_i^2 = b_1 c_1^2 + b_2 c_2^2 = \frac{1}{3}$$

and

$$\sum_{i,j} b_i a_{i,j} c_j = b_1(a_{1,1}c_1 + a_{1,2}c_2) + b_2(a_{2,1}c_1 + a_{2,2}c_2) = \frac{1}{6}$$

are satisfied. We find

$$\sum_i b_i c_i^2 = -2\mu(\mu - 1), \qquad \sum_{i,j} b_i a_{i,j} c_j = -\mu(\mu - 1).$$

The method has the third order if

$$-\mu(\mu - 1) = \frac{1}{6}$$

that gives us

$$\mu = \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{3}}\right).$$

d) We should compute the stability function

$$r(z) = 1 + z\mathbf{b}^T(I - zA)^{-1}\mathbf{1}.$$

We find

$$\det(I - zA) = 1 - 2z\mu + z^2\mu^2;$$

$$(I - zA)^{-1} = \frac{1}{\det(I - zA)} \begin{bmatrix} 1 - \left(1 + \frac{\sqrt{2}}{4}\right) z\mu & \frac{4-3\sqrt{2}}{4} z\mu \\ \frac{4+3\sqrt{2}}{4} z\mu & 1 - \left(1 - \frac{\sqrt{2}}{4}\right) z\mu \end{bmatrix}$$

$$r = 1 + z\frac{1 - 2z\mu + \frac{z}{2}}{1 - 2z\mu + z^2\mu^2} = \frac{1 + z(1 - 2\mu) + z^2(\mu^2 - 2\mu + \frac{1}{2})}{(1 - z\mu)^2} = \frac{P(z)}{Q(z)}$$

e) For A–stability we impose two conditions:

1. The polynomial Q must have roots $\mathbf{Re}\ z_i > 0$. The only root of multiplicity 2

$$z = \frac{1}{\mu}$$

satisfies this requirement if $\mu > 0$.

2. The equation

$$|r(iy)| \leq 1 \qquad \text{or} \qquad |P(iy)|^2 \leq |Q(iy)|^2$$

must be satisfied for any $y \in \mathbf{R}$. We obtain

$$|1 + iy(1 - 2\mu) - y^2(\mu^2 - 2\mu + \frac{1}{2})|^2 \leq |1 - iy\mu|^4$$

$$(y(1 - 2\mu))^2 + (1 - y^2(\mu^2 - 2\mu + \frac{1}{2}))^2 \leq (1 + y^2\mu^2)^2$$

$$(1 - 2\mu)^2 - 2(\mu^2 - 2\mu + \frac{1}{2}) + y^2(\mu^2 - 2\mu + \frac{1}{2})^2 \leq 2\mu^2 + y^2\mu^4$$

$$(\mu^2 - 2\mu + \frac{1}{2})^2 \leq \mu^4$$

Finally,

$$|\mu^2 - 2\mu + \frac{1}{2}| \leq \mu^2$$

If $\mu^2 - 2\mu + \frac{1}{2} \leq 0$ that is true for $\mu \in [\frac{2-\sqrt{2}}{2}, \frac{2+\sqrt{2}}{2}]$, then we get

$$-\mu^2 + 2\mu - 0.5 \leq \mu^2$$

This equation is always true so that for $\mu \in [\frac{2-\sqrt{2}}{2}, \frac{2+\sqrt{2}}{2}]$ the method is A–stable.

If $\mu^2 - 2\mu + 0.5 > 0$ that is true for $\mu \in (-\infty, \frac{2-\sqrt{2}}{2}) \cup (\frac{2+\sqrt{2}}{2}, \infty)$, then we get

$$\mu^2 - 2\mu + 0.5 \leq \mu^2.$$

This equation holds for $\mu \geq \frac{1}{4}$. Thus the method is A–stable for $\mu \in (\frac{1}{4}, \frac{2-\sqrt{2}}{2}) \cup (\frac{2+\sqrt{2}}{2}, \infty)$

Combining the obtained intervals of A–stability, we find out that the method is A–stable for $\mu \in [\frac{1}{4}, \infty)$.

Problem 2

a) Let us substitute the solution of the original continuous equation

$$u_t = u_{xx}$$

into the difference scheme

$$\frac{U_l^{n+2} - U_l^n}{2\Delta t} = \frac{U_{l+1}^{n+1} - 2U_l^{n+1} + U_{l-1}^{n+1}}{(\Delta x)^2}.$$

Using the Taylor expansion for the point $(x,t) = (x_l, t_{n+1})$, we find

$$\frac{u(x,t+\Delta t) - u(x,t-\Delta t)}{2\Delta t} - \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{(\Delta x)^2} =$$

$$= u_t + \frac{\Delta t^2}{3!}u_{ttt} + O(\Delta t^4) - \left(u_{xx} + 2\frac{\Delta x^2}{4!}u_{xxxx} + O(\Delta x^4)\right) = O(\Delta t^2, \Delta x^2) = O(\Delta x^2)$$

for $\Delta t = \mu \Delta x^2$. Thus, the scheme has the second order.

b) We substitute the solution of the form

$$U_l^n = \xi^n \exp(i\theta l)$$

into the scheme and obtain

$$\xi - \frac{1}{\xi} = 2\mu(\exp(i\theta) - 2 + \exp(i\theta l)) = -8\mu \sin^2\left(\frac{\theta}{2}\right).$$

Let us introduce the notation $a = 4\mu \sin^2\left(\frac{\theta}{2}\right) \geq 0$. The equation

$$\xi^2 + 2a\xi - 1 = 0$$

has two roots

$$\xi_1 = -a + \sqrt{1+a^2}, \qquad \xi_2 = -a - \sqrt{1+a^2}.$$

It is easy to show that $|\xi_1| \leq 1$ for any $a \geq 0$ and that $|\xi_2| \leq 1$ only for $a \leq 0$. It follows that the scheme is unstable for any $\mu > 0$.

Problem 3

a) Let us numerate the boundary points as shown

$x_{0,0}$	$x_{1,0}$	$x_{2,0}$	$x_{3,0}$	$x_{4,0}$
$x_{0,1}$	x_1	x_3	x_6	$x_{4,1}$
$x_{0,2}$	x_2	x_5	x_8	$x_{4,2}$
$x_{0,3}$	x_4	x_7	x_9	$x_{4,3}$
$x_{0,4}$	$x_{1,4}$	$x_{2,4}$	$x_{3,4}$	$x_{4,4}$

Then we obtain 9 equations for the internal mesh points

$$-4x_1 + x_2 + x_3 = -x_{0,1} - x_{1,0}$$

$$-4x_2 + x_1 + x_4 + x_5 = -x_{0,2}$$

$$-4x_3 + x_1 + x_5 + x_6 = -x_{2,0}$$

$$-4x_4 + x_2 + x_7 = -x_{0,3} - x_{1,4}$$

$$-4x_5 + x_2 + x_3 + x_7 + x_8 = 0$$

$$-4x_6 + x_3 + x_8 = -x_{3,0} - x_{4,1}$$

$$-4x_7 + x_4 + x_5 + x_9 = -x_{2,4}$$

$$-4x_8 + x_5 + x_6 + x_9 = -x_{4,2}$$

$$-4x_9 + x_7 + x_8 = -x_{3,4} - x_{4,3}$$

These equations can be presented in the matrix form

$$BU = F$$

with

$$B = \begin{pmatrix} -4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -4 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -4 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -4 \end{pmatrix}$$

and

$$F = (-x_{0,1} - x_{1,0}; -x_{0,2}; -x_{2,0}; -x_{0,3} - x_{1,4}; 0; -x_{3,0} - x_{4,1}; -x_{2,4}; -x_{4,2}; -x_{3,4} - x_{4,3})^T$$

b) We should change the numeration of the internal mesh points from

x_1	x_3	x_6
x_2	x_5	x_8
x_4	x_7	x_9

to

x_1	x_4	x_7
x_2	x_5	x_8
x_3	x_6	x_9

This renumeration is realized as a transformation with the permutation matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

This transformation essentially interchanges two pairs of elements: $x_3 \leftrightarrow x_4$ and $x_6 \leftrightarrow x_7$.

The matrix $A = PBP^{-1}$ is the standard Poisson solver which is nonsingular (proved in the book). From nonsingularity of the matrices A and P it follows that the matrix B is also nonsingular.