NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for
TMA4212 Numerical solution of differential equations with difference methods

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Permitted examination support material: C: Approved simple pocket calculator is allowed. The text book by *Strikwerda*, the book by *Süli and Mayers*, and the official note of the TMA4212 course (98 pages) are allowed. Photo copies on 2D finite elements (4 pages) are allowed. Rottman is allowed. Old exams with solutions are *not* allowed.

Language: English

Number of pages: 7

Number pages enclosed: 2

Checked by:

Date Signature

The learning outcome has been published on the course webpage and on the official description of the course. The seven learning goals **L1** to **L7** are reported in the appendix. Learning outcome **L6**, **L3** and to some extent **L4** and **L5** have been tested through the project work. We here test further the achievement of **L4** and **L5** as well as **L1, L2, L7**.

Problem 1 (**L1**)
The wave equation in one space dimension can be written in the form

$$\mathbf{u}_t + A\mathbf{u}_x = 0$$

where

$$\mathbf{u} := \begin{bmatrix} v(x,t) \\ w(x,t) \end{bmatrix}, \qquad A := \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Consider the following finite difference method for this problem

$$\begin{aligned} V_j^{k+1} &= V_j^k + \frac{1}{2}p\,(W_{j+1}^k - W_{j-1}^k), \\ W_j^{k+1} &= W_j^k + \frac{1}{2}p\,(V_{j+1}^{k+1} - V_{j-1}^{k+1}), \end{aligned}$$

with $p = \frac{\Delta t}{\Delta x}$.

a) Find the leading error term of the local truncation error for this method.
Solution. The Taylor expansion and using $v_t = w_x$, $w_t = v_x$, $w_{xx} = v_{tx}$, $w_{xx} = w_{tt}$ gives the following local truncation error

$$\begin{aligned} \tau_j^k(v) &= \frac{1}{2}\Delta t \partial_x^2 v_j^k - \frac{1}{6}\Delta x^2 \partial_x^3 w_j^k + \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^3) \\ \tau_j^k(w) &= -\frac{1}{2}\Delta t \partial_x^2 w_j^k - \frac{1}{6}\Delta x^2 \partial_x^3 v_j^k + \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^3), \end{aligned}$$

and the first two terms in each expression are the leading error term of the local truncation error.

b) Perform a von Neumann stability analysis for this numerical method.
Solution. We set $V_j^k = \xi_k e^{i\beta x_j}$ and $W_j^k = \eta_k^{i\beta x_j}$ and substitute in the formula for the numerical method, we get

$$\begin{aligned} \xi_{k+1} &= \xi_k + i\,p\,\eta_k \sin(\beta\Delta x) \\ \eta_{k+1} &= \eta_k + i\,p\,\xi_{k+1} \sin(\beta\Delta x) \end{aligned}$$

solving for ξ_{k+1} and η_{k+1} we get

$$\begin{bmatrix} \xi_{k+1} \\ \eta_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & ip\sin(\beta\Delta x) \\ ip\sin(\beta\Delta x) & 1-p^2\sin^2(\beta\Delta x) \end{bmatrix} \cdot \begin{bmatrix} \xi_k \\ \eta_k \end{bmatrix} = G \cdot \begin{bmatrix} \xi_k \\ \eta_k \end{bmatrix}$$

We need $\rho(G) \leq 1$ for stability. We see that $\det(G) = 1$. Then $\rho(G) = 1$ if the two eigenvalues of G are such that $\lambda_1 = \bar{\lambda}_2$, and $\rho(G) > 1$ otherwise. The condition for stability is then

$$\text{Tr}(G)^2 - 4\det(G) \leq 0 \Rightarrow p^2\sin(\beta\Delta x)^2(p\sin(\beta\Delta x)+2)(p\sin(\beta\Delta x)-2) \leq 0.$$

The method is Von Neumann stable if $|p|\,|\sin(\beta\Delta x)| \leq 2$ for all β, i.e. $|p| \leq 2$.

Problem 2 (**L1, L4, L5**) The function $u = u(x,y)$ satisfies the equation $u_{xx} + u_{yy} + f(x,y) = 0$ in the sector of the circle defined by $0 \leq x^2 + y^2 \leq 1$, $0 \leq y \leq x$. A zero Neumann condition is given on the boundary $x = y$ and homogeneous Dirichlet conditions on the rest of the boundary. Using a uniform square mesh of size $\Delta x = \Delta y = \frac{1}{3}$ leads to a system of linear equations of the form $A\mathbf{u} = \mathbf{b}$.

a) Construct explicitly the elements of the matrix A.
Solution. The domain and the grid can be seen in figure 1. There are three unknowns (three components in the vector $\mathbf{u} = [U_1, U_2, U_3]^T$ and they correspond to the numerical approximations of u in the two nodes of the grid along the line $x = y$ (points of coordinates $(\frac{1}{3}, \frac{1}{3})$ and $(\frac{2}{3}, \frac{2}{3})$) and the only internal node of the grid (coordinates $(\frac{2}{3}, \frac{1}{3})$). The normal vector along the boundary $x = y$ is $\mathbf{n} = \sqrt{2}[1, -1]$, and the normal derivative is $\partial_{\mathbf{n}} u = \mathbf{n}^T \nabla u$. So the zero Neumann boundary conditions can be expressed as

$$u_x = u_y,$$

along $x = y$.

We order the unknowns is shown in the figure. We need to use finite differences with variable step-sizes. The equations we obtain are

$$\begin{aligned} U_2 - 2U_1 &= 0 \\ \frac{2}{0.6095}\left(\frac{-U_2}{0.2761} - \frac{U_2 - U_1}{1/3}\right) + \frac{U_3 - 2U_2 + 0}{(1/3)^2} &= -f_2 \\ \frac{U_3}{0.0787} + \frac{U_3 - U_2}{1/3} &= 0 \end{aligned}$$

from these we can deduce the matrix A which in this case is 3×3.

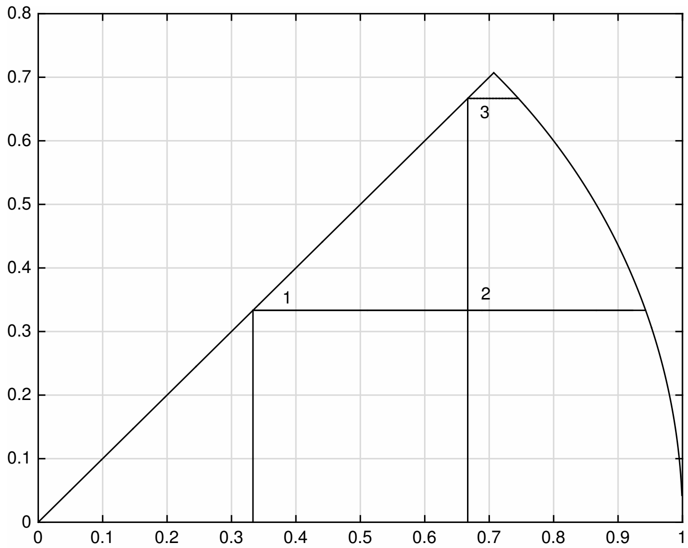

Figure 1: Grid for problem 2 a).

b) Transform the problem in polar coordinates and construct the matrix of a similar system of linear equations. Use four unknowns and a 4×4 matrix A. **Solution**. We built the grid in polar coordinates as given in figure 2. In this way and there are four unknowns corresponding to two internal nodes and two boundary nodes (along the line $x = y$). In vector notation this amounts to $\mathbf{u} = [U_1, U_2, U_3, U_4]^T$ and we order the unknowns is as shown in the figure. The four corresponding equations are

$$\begin{aligned} \frac{1}{(1/3)^2}(U_2 - 2U_1) + \frac{9}{2}U_2 + 9\frac{U_3 - 2U_1}{\pi^2}64 &= -f_1 \\ \frac{1}{(1/3)^2}(-2U_2 + U_1) - \frac{9}{4}U_1 + \frac{9}{4}\frac{U_4 - 2U_2}{\pi^2}64 &= -f_2 \\ U_3 - U_1 &= 0 \\ U_4 - U_2 &= 0 \end{aligned}$$

here $\Delta r = \frac{1}{3}$. From these equations we can deduce the matrix A which in this case is 4×4.

Problem 3 (**L1, L4**) Consider the following finite difference method

$$\frac{U_m^{n+1} - U_m^n}{\Delta t} = \frac{U_{m-1}^{n+1} - 2U_m^{n+1} + U_{m+1}^{n+1}}{\Delta x^2}, \quad m = 1, \ldots, M, \quad n \geq 0, \quad \Delta x = \frac{1}{M+1}$$

for the heat equation

$$u_t = u_{xx}, \quad x \in [0, 1], \quad t > 0,$$

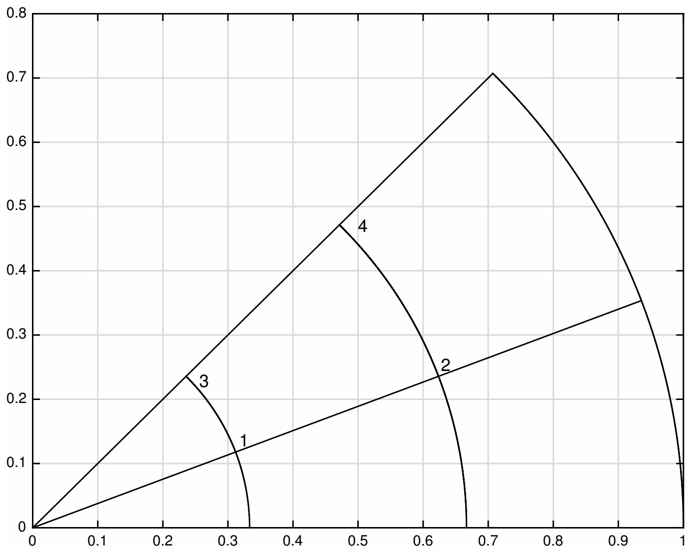

Figure 2: Grid for problem 2 b).

with homogeneous Dirichlet boundary conditions and initial condition $u(x,0) = f(x)$. We assume the solution is sufficiently regular, and that the following bound for the local truncation error τ_m^n holds

$$|\tau_m^n| \leq A(\Delta t + \Delta x^2), \quad \forall m, n$$

where A is a constant not depending on Δt and Δx.

Show that the method converges for all values of $r = \frac{\Delta t}{\Delta x^2}$ and finite time T, without using the Lax equivalence theorem.

Solution. We use an approach similar to the one of chapter 5.4 in the note. Consider $u_m^n := u(x_m, t_n)$ the local truncation error τ_m^n satisfies the equation

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} = \left[\frac{u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}}{\Delta x^2}\right] + \tau_m^n$$

subtracting the equation defining the method from this equation we obtain the equation for the error $e_m^n = u_m^n - U_m^n$,

$$e_m^{n+1} = e_m^n + r\left[e_{m-1}^{n+1} - 2e_m^{n+1} + e_{m+1}^{n+1}\right] + \Delta t\, \tau_m^n$$

and

$$(1+2r)e_m^{n+1} = e_m^n + r\left[e_{m-1}^{n+1} + e_{m+1}^{n+1}\right] + \Delta t\, \tau_m^n$$

taking absolute values

$$(1+2r)|e_m^{n+1}| \leq |e_m^n| + r\left[|e_{m-1}^{n+1}| + |e_{m+1}^{n+1}|\right] + |\Delta t\, \tau_m^n|.$$

Defining $E^n := \max_m |e_m^n|$ and maximizing on the right hand side we get

$$(1+2r)|e_m^{n+1}| \leq E^n + 2rE^{n+1} + \Delta t\, A(\Delta t^2 + \Delta x^2), \quad \forall m.$$

Since the inequality holds for all m it holds also for the maximum over m and so

$$(1+2r)E^{n+1} \leq (E^n + 2rE^{n+1}) + \Delta t\, A(\Delta t + \Delta x^2)$$

$$E^{n+1} \leq E^n + \Delta t\, A(\Delta t + \Delta x^2)$$

so

$$E^n \leq An(\Delta t^2 + \Delta t \Delta x^2) \leq TA(\Delta t + \Delta x^2), \quad n\Delta t \leq T.$$

So

$$\lim_{\Delta x \to 0, \Delta t \to 0} E^n = 0.$$

Problem 4 (**L2, L7**)
Consider the boundary value problem

$$\begin{aligned} -\frac{d}{dx}\left((x+1)\frac{du}{dx}\right) &= f(x), \quad x \in [0,4] \\ u(0) &= 0, \\ u(4) &= 0. \end{aligned} \tag{1}$$

We will solve this problem using the finite element method.

a) State the weak formulation of the problem.

Solution: By multiplying with a test function v and integrating over the domain we get

$$\begin{aligned} \int_0^4 -\frac{d}{dx}\left((x+1)\frac{du}{dx}\right) v dx &= \int_0^4 fv dx \\ \int_0^4 (x+1)u'v'dx - [(x+1)u'v]_0^4 &= \int_0^4 fv dx \\ \int_0^4 (x+1)u'v'dx &= \int_0^4 fv dx \\ A(u,v) &= l(v) \end{aligned}$$

since $v(0) = v(4) = 0$. The weak formulation is "Find $u \in H_0^1(0,4)$ such that $A(u,v) = l(v) \quad \forall v \in H_0^1(0,4)$" . We sometimes write $l(v) = \langle f, v \rangle$.

Consider the approximation space of piecewise linear polynomials (hat functions) over the grid nodes $x_0 = 0, x_1 = 2, x_2 = 3, x_3 = 4$.

b) Write the discrete problem (Galerkin Method) and specify the function space S_0^h by writing explicit expressions for the basis functions $\varphi_i(x)$.

Solution: The Galerking method reads "Find $u_h \in S_0^h$ such that $A(u_h, v_h) = l(v_h) \quad \forall v_h \in S_0^h$" . Where $S_0^h = \left\{v^h \in H_0^1(0,4) | v^h = \sum_{i=1}^{2} v_i \varphi_i\right\}$. The basis functions are

$$\varphi_1(x) = \begin{cases} x/2 & x \in [0,2] \\ 3-x & x \in [2,3] \\ 0 & \text{else} \end{cases}$$

$$\varphi_2(x) = \begin{cases} x-2 & x \in [2,3] \\ 4-x & x \in [3,4] \\ 0 & \text{else} \end{cases}$$

The Galerkin method allows us to find the numerical solution of the differential equation (1) by solving a linear system of equations

$$Au = b.$$

c) What is the definition of the elements A_{ij} of the A-matrix and b_i of the b-vector for this differential equation. What is the size of A and b for this choice of S_0^h. Compute all elements of the A-matrix using the linear hat functions from b).

Solution:

$$\begin{aligned} A_{ij} = A(\varphi_i, \varphi_j) &= \int_0^4 (x+1)\varphi_i' \varphi_j' \, dx \\ b_i = l(\varphi_i) &= \int_0^4 \varphi_i f \, dx \end{aligned}$$

$A \in \mathbb{R}^{2\times 2}$ and $b \in \mathbb{R}^2$. To compute A_{ij}, we first note that the derivative of the basis functions are

$$\varphi_1'(x) = \begin{cases} 1/2 & x \in [0,2] \\ -1 & x \in [2,3] \\ 0 & \text{else} \end{cases}$$

$$\varphi_2'(x) = \begin{cases} 1 & x \in [2,3] \\ -1 & x \in [3,4] \\ 0 & \text{else} \end{cases}$$

$$\begin{array}{lcl}
A(\varphi_1,\varphi_1) = & \int_0^2 (x+1)(\frac{1}{2})^2 dx + \int_2^3 (x+1)(-1)^2 dx & = 9/2 \\
A(\varphi_1,\varphi_2) = & \int_2^3 (x+1)(-1)(1) dx & = -7/2 \\
A(\varphi_2,\varphi_2) = & \int_2^3 (x+1)(1)^2 dx + \int_3^4 (x+1)(-1)^2 dx & = 8
\end{array}$$

Noting that A is symmetric we then have

$$A = \frac{1}{2}\begin{bmatrix} 9 & -7 \\ -7 & 16 \end{bmatrix}$$

Appendix Laplacian in polar coordinates $x = r\cos(\varphi)$, $y = r\sin(\varphi)$

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}$$

Learning outcome:

Knowledge	**L1**	Understanding of error analysis of difference methods: consistency, stability, convergence of difference schemes.
	L2	Understanding of the basics of the finite element method.
Skills	**L3**	Ability to choose and implement a suitable discretization scheme given a particular PDE, and to design numerical tests in order to verify the correctness of the code and the order of the method.
	L4	Ability to analyze the chosen discretization scheme, at least for simple PDE-test problems.
	L5	Ability to attack the numerical linear algebra challenges arising in the numerical solution of PDEs.
General competence	**L6**	Ability to present in oral and written form the numerical and analytical results obtained in the project work.
	L7	Ability to apply acquired mathematical knowledge in linear algebra and calculus to achieve the other goals of the course.