NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for
TMA4212 Numerical solution of differential equations with difference methods

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Examination time (from–to): 09:00-13:00

Permitted examination support material: C: Approved simple pocket calculator is allowed. The text book by *Strikwerda*, the book by *Süli and Mayers*, and the official note of the TMA4212 course (98 pages) are allowed. Photo copies on 2D finite elements (4 pages) are allowed. Rottman is allowed. Old exams with solutions are *not* allowed.

Language: English

Number of pages: 3

Number pages enclosed: 2

Checked by:

Date Signature

The learning outcome has been published on the course webpage and on the official description of the course. The seven learning goals **L1** to **L7** are reported in the appendix. Learning outcome **L6**, **L3** and to some extent **L4** and **L5** have been tested through the project work. We here test further the achievement of **L4** and **L5** as well as **L1, L2, L7**.

Problem 1 (**L1**)
The wave equation in one space dimension can be written in the form

$$\mathbf{u}_t + A\mathbf{u}_x = 0$$

where

$$\mathbf{u} := \begin{bmatrix} v(x,t) \\ w(x,t) \end{bmatrix}, \qquad A := \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Consider the following finite difference method for this problem

$$\begin{aligned} V_j^{k+1} &= V_j^k + \frac{1}{2} p \, (W_{j+1}^k - W_{j-1}^k), \\ W_j^{k+1} &= W_j^k + \frac{1}{2} p \, (V_{j+1}^{k+1} - V_{j-1}^{k+1}), \end{aligned}$$

with $p = \frac{\Delta t}{\Delta x}$.

a) Find the leading error term of the local truncation error for this method.

b) Perform a von Neumann stability analysis for this numerical method.

Problem 2 (**L1, L4, L5**) The function $u = u(x,y)$ satisfies the equation $u_{xx} + u_{yy} + f(x,y) = 0$ in the sector of the circle defined by $0 \leq x^2 + y^2 \leq 1$, $0 \leq y \leq x$. A zero Neumann condition is given on the boundary $x = y$ and homogeneous Dirichlet conditions on the rest of the boundary. Using a uniform square mesh of size $\Delta x = \Delta y = \frac{1}{3}$ leads to a system of linear equations of the form $A\mathbf{u} = \mathbf{b}$.

a) Construct explicitly the elements of the matrix A.

b) Transform the problem to polar coordinates and construct the matrix of a similar system of linear equations. Use four unknowns and a 4×4 matrix A.

Problem 3 (**L1, L4**) Consider the following finite difference method

$$\frac{U_m^{n+1} - U_m^n}{\Delta t} = \frac{U_{m-1}^{n+1} - 2U_m^{n+1} + U_{m+1}^{n+1}}{\Delta x^2}, \quad m = 1, \ldots, M, \quad n \geq 0, \quad \Delta x = \frac{1}{M+1}$$

for the heat equation

$$u_t = u_{xx}, \quad x \in [0,1], \quad t > 0,$$

with homogeneous Dirichlet boundary conditions and initial condition $u(x,0) = f(x)$. We assume the solution is sufficiently regular, and that the following bound for the local truncation error τ_m^n holds

$$|\tau_m^n| \leq A(\Delta t^2 + \Delta x^2), \quad \forall m, n$$

where A is a constant not depending on Δt and Δx.

Show that the method converges for all values of $r = \frac{\Delta t}{\Delta x^2}$ and finite time T, without using the Lax equivalence theorem.

Problem 4 (**L2, L7**)
Consider the boundary value problem

$$\begin{aligned} -\frac{d}{dx}\left((x+1)\frac{du}{dx}\right) &= f(x), \quad x \in [0,4] \\ u(0) &= 0, \\ u(4) &= 0. \end{aligned} \tag{1}$$

We will solve this problem using the finite element method.

a) State the weak formulation of the problem.

Consider the approximation space of piecewise linear polynomials (hat functions) over the grid nodes $x_0 = 0, x_1 = 2, x_2 = 3, x_3 = 4$.

b) Write the discrete problem (Galerkin method) and specify the function space S_0^h by writing explicit expressions for the basis functions $\varphi_i(x)$.

The Galerkin method allows us to find the numerical approximation of the differential equation (1) by solving a linear system of equations

$$Au = b.$$

c) What is the definition of the elements A_{ij} of the A-matrix and b_i of the b-vector for this differential equation. What is the size of A and b for this choice of S_0^h. Compute all elements of the A-matrix using the linear hat functions from b).

Appendix Laplacian in polar coordinates $x = r\cos(\varphi)$, $y = r\sin(\varphi)$

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}$$

Learning outcome:

Knowledge	**L1**	Understanding of error analysis of difference methods: consistency, stability, convergence of difference schemes.
	L2	Understanding of the basics of the finite element method.
Skills	**L3**	Ability to choose and implement a suitable discretization scheme given a particular PDE, and to design numerical tests in order to verify the correctness of the code and the order of the method.
	L4	Ability to analyze the chosen discretization scheme, at least for simple PDE-test problems.
	L5	Ability to attack the numerical linear algebra challenges arising in the numerical solution of PDEs.
General competence	**L6**	Ability to present in oral and written form the numerical and analytical results obtained in the project work.
	L7	Ability to apply acquired mathematical knowledge in linear algebra and calculus to achieve the other goals of the course.