NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for
TMA4212 Numerical solution of differential equations with difference methods

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Permitted examination support material: C: Approved simple pocket calculator is allowed. The text book by Strikwerda, the book by Süli and Mayers, and the official note of the TMA4212 course (98 pages) are allowed. Photo copies on 2D finite elements (4 pages) are allowed. Rottman is allowed. Old exams with solutions are not allowed.

Language: English

Number of pages: 11

Number pages enclosed: 2

Checked by:

Date Signature

The learning outcome has been published on the course webpage and on the official description of the course. The seven learning goals **L1** to **L7** are reported in the appendix. Learning outcome **L6**, **L3** and to some extent **L4** and **L5** have been tested through the project work. We here test further the achievement of **L4** and **L5** as well as **L1, L2, L7**.

Guidelines for the marking

Problem 1 a: $A(u, v) = \ell(v)$ right subspace of u 6, right ℓ and right A 8, right $H_0^1(0, 1)$ for test function v10.
Problem 1 b: $A(u, v) = \ell(v)$ 6, right ℓ and right A 8, right subspace of $H_0^1(0, 1)$ 10.
Problem 1 c: $A(u, v) = \ell(v)$ 6, right ℓ and right A 8, right subspace of $H_0^1(0, 1)$ 10.
Problem 2 a: right figure 6, right adjustment of the five points formula 8, right system 10.
Problem 2 b: A SPD with right explanation 6, CG yes 8, CG yes right explanation 10.
Problem 3 a: characteristics right explanation 6, characteristics meet x-axis in the interval $(0, 1)$ 10.
Problem 3 b: Right use of the method 6, quite right computation only minor mistakes 8, correct 10.
Problem 4 a: right LTE 6, bound of A?1 in 2-norm and bound of the error 8, invertibility of A independently of n 10.
Problem 4 b: right 10, minor computation error 8, major computational error 6.
Problem 4 c: First or second part 4, last part assuming the two first parts 6, first and second part 6, first or second part and last part 8, all parts 10.

Problem 1 (**L2, L7**)
Given $f \in L^2(0, 1)$, state the weak formulation of each of the following boundary value problems on the interval $(0, 1)$:

a)

$$-u'' + u = f(x), \quad u(0) = 0, \quad u(1) = 0;$$

Solution
The Galerkin formulation is:

Find $u \in H^1(0, 1)$, with $u(0) = u(1) = 0$ such that

$$A(u, v) = \ell(v)$$

for all $v \in H^1(0,1)$ with $v(0) = v(1) = 0$, where

$$A(u,v) := \int_0^1 (u'v' + uv)\,dx, \quad \ell(v) := \int_0^1 fv\,dx.$$

Points: $A(u,v) = \ell(v)$ 6, right ℓ and right A 8, right $H_0^1(0,1)$ 10.

b)

$$-u'' + u = f(x), \quad u(0) = 0, \quad u'(1) = 1;$$

Solution
The Galerkin formulation is:

Find $u \in H^1(0,1)$, with $u(0) = 0$ such that

$$A(u,v) = \ell(v)$$

for all $v \in H^1(0,1)$ with $v(0) = 0$, where

$$A(u,v) := \int_0^1 (u'v' + uv)\,dx, \quad \ell(v) := \int_0^1 fv\,dx + v(1).$$

Points: $A(u,v) = \ell(v)$ 6, right ℓ and right A 8, right subspace of $H_0^1(0,1)$ 10.

c)

$$-u'' + u = f(x), \quad u(0) = 0, \quad u(1) + u'(1) = 2.$$

Solution
The Galerkin formulation is:

Find $u \in H^1(0,1)$, with $u(0) = 0$ such that

$$A(u,v) = \ell(v)$$

for all $v \in H^1(0,1)$ with $v(0) = 0$, where

$$A(u,v) := \int_0^1 (u'v' + uv)\,dx + u(1)v(1), \quad \ell(v) := \int_0^1 fv\,dx + 2v(1).$$

Points: $A(u,v) = \ell(v)$ 6, right ℓ and right A 8, right subspace of $H_0^1(0,1)$ 10.

Problem 2 (**L1, L4, L5, L7**)

a) Construct explicitly the system of linear equations obtained from approximating Poisson's equation

$$u_{xx} + u_{yy} + f(x, y) = 0$$

in the region defined by $x \geq 0$, $y \geq 0$, $x^2 + y \leq 1$. The boundary conditions are $u(x, 0) = p(x)$, $u(0, y) = q(y)$, and $u(x, 1 - x^2) = r(x)$, where p, q, r and f are given functions.

Use a grid of size $\Delta x = \frac{1}{3}$ and $\Delta y = \frac{1}{2}$. Use the five point formula. Use variable step-size near the right boundary.
(Show that you can apply the method correctly. You do not need to rearrange the terms in the form $AU = b$.)

Solution
The domain and the grid are depicted in Figure 1, with We use the grid points $x_i = \Delta x \cdot i$, $(i = 1, 2)$ $y_1 = \Delta y$, $\Delta x = \frac{1}{3}$, $\Delta y = \frac{1}{2}$. Starting from left to right we number the unknowns as follows $U_1 \approx u(x_1, y_1)$, $U_2 \approx u(x_2, y_1)$.

The system of equations in the unknowns U_1 and U_2 is

$$\begin{aligned}
\left(\frac{U_2-U_1}{\Delta x} - \frac{U_1-q(y_1)}{\Delta x}\right)\frac{1}{\Delta x} + \left(\frac{r(x_1)-U_1}{\Delta y_1} - \frac{U_1-p(x_1)}{\Delta y}\right)\frac{2}{\Delta y_1+\Delta y} &= -f(x_1, y_1) \\
\left(\frac{r(\sqrt{1-y_1})-U_2}{\Delta x_2} - \frac{U_2-U_1}{\Delta x}\right)\frac{2}{\Delta x_2+\Delta x} + \left(\frac{r(x_2)-U_2}{\Delta y_2} - \frac{U_2-p(x_2)}{\Delta y}\right)\frac{2}{\Delta y_2+\Delta y} &= -f(x_2, y_1)
\end{aligned} \tag{1}$$

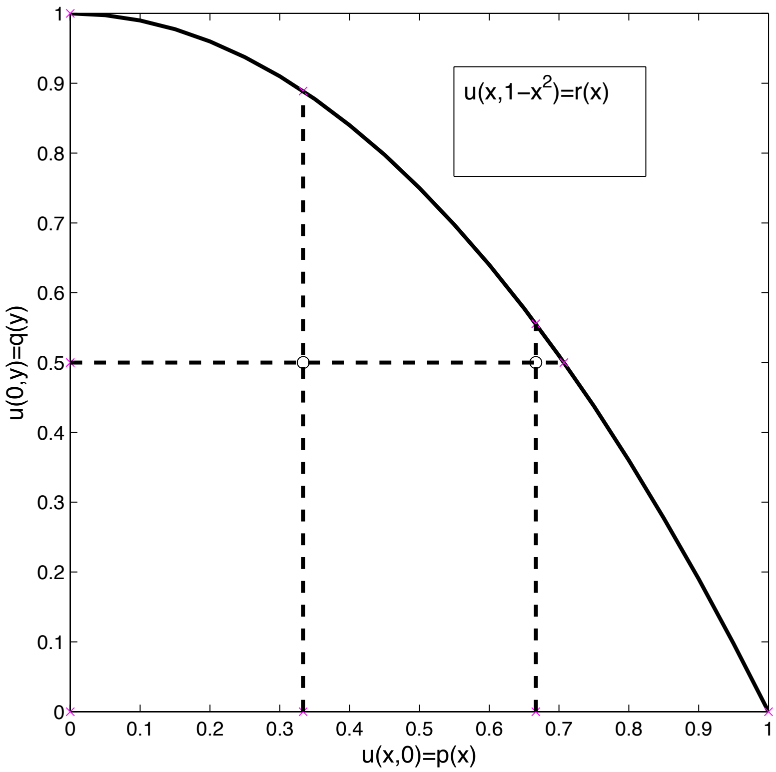
$$\Delta y_1 = 1 - x_1^2 - y_1, \quad \Delta x_2 = \sqrt{1 - y_1} - x_2, \quad \Delta y_2 = 1 - x_2^2 - y_1.$$

Points: right figure 6, right adjustment of the five points formula 8, right system 10.

b) Consider a linear system $Ax = b$ with the following structure:

$$A = \begin{bmatrix} B_m & \tilde{I}_m & O & \dots & O \\ \tilde{I}_m^T & B_{m-1} & \tilde{I}_{m-1} & \ddots & \vdots \\ O & \tilde{I}_{m-1}^T & \ddots & \ddots & O \\ \vdots & \ddots & & \ddots & \tilde{I}_2 \\ O & \dots & O & \tilde{I}_2^T & B_1 \end{bmatrix}$$

Figure 1: Domain and grid.



where B_k is $k \times k$ tridiagonal with $\alpha > 4$ on the diagonal, $-\tilde{I}_k$ is $k \times (k-1)$ and it is obtained by removing the last column from the $k \times k$ identity matrix i.e.

$$B_k = \begin{bmatrix} \alpha & -1 & 0 & \dots & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & & & & \\ \vdots & & & \ddots & -1 \\ 0 & \dots & 0 & -1 & \alpha \end{bmatrix}, \qquad \tilde{I}_k = -\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \end{bmatrix}.$$

Is A is symmetric and positive definite? Explain your answer. Can you use the Conjugate Gradient method to approximate the solution of this linear system iteratively?

Solution

This matrix is symmetric and positive definite. The symmetry comes from noticing that

$$A^T = \begin{bmatrix} B_m^T & (\tilde{I}_m^T)^T & O & \dots & O \\ \tilde{I}_m^T & B_{m-1}^T & (\tilde{I}_{m-1}^T)^T & \ddots & \vdots \\ O & \tilde{I}_{m-1}^T & \ddots & \ddots & O \\ \vdots & \ddots & & \ddots & (\tilde{I}_2^T)^T \\ O & \dots & O & \tilde{I}_2^T & B_1^T \end{bmatrix}$$

with $(\tilde{I}_k^T)^T = \tilde{I}_k$ and $B_k = B_k^T$ gives $A^T = A$. The eigenvalues are all real since the matrix is symmetric. The matrix is positive definite because on each row there are at most 4 off-diagonal elements which are different from zero (equal to -1). The sum of the absolute values of the off-diagonal elements of the matrix is strictly less than α so by the Gershgorin theorem all eigenvalues are positive.

Since A is symmetric and positive definite, the Conjugate Gradient method is well defined and can be applied to approximate the solution to this problem.
Points: A SPD with right explanation 6, CG yes 8, CG yes right explanation 10.

Problem 3 (L1, L4, L7)

a) Consider the equation

$$u_t + a\, u_x = 0, \quad 0 \le x \le 1, \quad t \ge 0,$$

when $a = a(x) = x - \frac{1}{2}$, with a given initial function $u(x,0)$ for $0 \le x \le 1$. Show that the characteristics are

$$x(t) = e^t\,(x_0 - \frac{1}{2}) + \frac{1}{2}.$$

Explain why you do not need to impose any boundary conditions.

Solution
By differentiation we see that the given $x(t)$ is the solution of the characteristic equation $\dot{x} = a(x(t))$, with $x(0) = x_0$ in fact

$$\dot{x} = e^t(x_0 - \frac{1}{2}) = x(t) - \frac{1}{2}, \quad x(0) = x_0.$$

Solving for x_0 from $x(t) = e^t\,(x_0 - \frac{1}{2}) + \frac{1}{2}$ we get

$$x_0 = e^{-t}(x(t) - \frac{1}{2}) + \frac{1}{2}$$

which is the point in which the characteristic curve intersects the x-axis in the (x,t)-plane. If $0 \le x(t) \le 1$ and $t > 0$,

$$0 \le x_0 = e^{-t}(x(t) - \frac{1}{2}) + \frac{1}{2} \le x(t) - \frac{1}{2} + \frac{1}{2} \le x(t) \le 1$$

and x_0 is in the interval $[0,1]$. So to obtain the solution in (x,t) which is $u(x,t) = u(e^{-t}(x - \frac{1}{2}) + \frac{1}{2}, 0)$ it suffices to know the initial function $u(x,0)$ on the interval $(0,1)$ and no boundary conditions are necessary.

One can also note that for all $s \le t$ we have

$$x(s) = e^s(x_0 - \frac{1}{2}) + \frac{1}{2} = e^s e^{-t}(x(t) - \frac{1}{2}) + \frac{1}{2},$$

$$0 \le e^{s-t}(x(t) - \frac{1}{2}) + \frac{1}{2} \le x(t) \le 1,$$

so

$$0 \le x(s) \le x(t) \le 1, \qquad \forall s \le t,$$

i.e. the characteristics do not leave the domain $0 \le x \le 1$, $t \ge 0$ where the solution of the PDE is sought.
Points: characteristics right explanation 6, characteristics meet x-axis in the interval $(0,1)$ 10.

b) The upwind method applied to the equation is

$$\begin{aligned} \frac{U_i^{n+1}-U_i^n}{k} + \left(x_i - \tfrac{1}{2}\right) \frac{U_i^n - U_{i-1}^n}{h} &= 0, \quad x_i > \tfrac{1}{2} \\ \frac{U_i^{n+1}-U_i^n}{k} + \left(x_i - \tfrac{1}{2}\right) \frac{U_{i+1}^n - U_i^n}{h} &= 0, \quad x_i < \tfrac{1}{2}, \end{aligned}$$

and we are considering a uniform mesh $x_i = i\,h$, $i = 0, \dots, N$, with k the temporal step-size.

Consider the initial function $u(x,0) = |x - \frac{1}{2}|$, $N = 4$ and $k = h$. Compute the error of the method in $x = 1/4$ and $x = 3/4$ and at time $t = k$. Recall that $u(x,t) = u(e^{-t}(x - \frac{1}{2}) + \frac{1}{2}, 0)$.

Solution

One obtains

$$u\left(\frac{1}{4}, k\right) = \left| e^{-k}\left(\frac{1}{4} - \frac{1}{2}\right)\right| = \left| -\frac{1}{4}e^{-k}\right|,$$

and

$$u\left(\frac{3}{4}, k\right) = \left| e^{-k}\left(\frac{3}{4} - \frac{1}{2}\right)\right| = \left| \frac{1}{4}e^{-k}\right|.$$

For $k = h = \frac{1}{4}$ we have $u\left(\frac{1}{4}, \frac{1}{4}\right) = u\left(\frac{3}{4}, \frac{1}{4}\right) = 0.194700$.

For $x_1 = \frac{1}{4} < \frac{1}{2}$ the method gives

$$\frac{U_1^1 - U_1^0}{k} + \left(\frac{1}{4} - \frac{1}{2}\right)\frac{U_2^0 - U_1^0}{h} = 0,$$

and

$$U_1^1 = U_1^0 + \frac{1}{4}\left(U_2^0 - U_1^0\right) = \frac{3}{16}.$$

For $x_3 = \frac{3}{4} \geq \frac{1}{2}$ the method gives

$$\frac{U_3^1 - U_3^0}{k} + \left(\frac{3}{4} - \frac{1}{2}\right)\frac{U_3^0 - U_2^0}{h} = 0,$$

and

$$U_3^1 = U_3^0 - \frac{1}{4}(U_3^0 - U_2^0) = \frac{3}{16}$$

the error is in both points

$$U_1^1 - u\left(\frac{1}{4}, \frac{1}{4}\right) = \frac{3}{16} - 0.194700 = U_3^1 - u\left(\frac{3}{4}, \frac{1}{4}\right) = -0.007200.$$

Points: Right use of the method 6, quite right computation only minor mistakes 8, correct 10.

Problem 4 (**L1, L4, L7**)
Consider the boundary value problem

$$-y'' + a^2 y = 0, \quad y(-1) = 1, \qquad y(1) = 1.$$

Consider the following difference approximation to the given problem

$$-\frac{Y_{j-1} - 2Y_j + Y_{j+1}}{h^2} + a^2 Y_j = 0, \qquad j = 1, 2, \dots, n-1, \quad Y_0 = Y_n = 1.$$

a) Find the local truncation error for this method. Then prove convergence in the 2-norm.

Solution
Replacing Y_j with $y(x_j)$ in the equation for the method gives the values of the local truncation error

$$-\frac{y(x_{j-1}) - 2y(x_j) + y(x_{j+1})}{h^2} + a^2 y(x_j) = \tau_j, \qquad j = 1, 2, \dots, n-1, \quad y(x_0) = y(x_n) = 1,$$

and using Taylor expansion we have

$$\frac{y(x_{j-1}) - 2y(x_j) + y(x_{j+1})}{h^2} = y''(x_j) + \frac{h^2}{12} y^{IV}(x_j) + \mathcal{O}(h^4),$$

which substituted into the equation for τ_j gives

$$\tau_j = -\frac{h^2}{12} y^{IV}(x_j) + \mathcal{O}(h^4).$$

Subtracting the equation for the local truncation error from the equation for the method we obtain the equation for the error $e_j := Y_j - y(x_j)$

$$-\frac{e_{j-1} - 2e_j + e_{j+1}}{h^2} + a^2 e_j = -\tau_j, \qquad j = 1, 2, \dots, n-1, \quad e_0 = e_n = 0.$$

Which in the form of a linear system is

$$Ae = \tau$$

with $e = [e_1, \dots, e_{n-1}]^T$, $\tau = [\tau_1, \dots, \tau_{n-1}]^T$. The matrix A is $(n-1) \times (n-1)$ tridiagonal with $2 + a^2$ on the main diagonal and -1 on the first sub-diagonal and on the first super-diagonal. This matrix is symmetric and positive definite for all n and therefore invertible, we thus get

$$e = A^{-1}\tau$$

and

$$\|e\|_2 \leq \|A^{-1}\|_2 \|\tau\|_2.$$

The norm $\|A^{-1}\|_2$ is bounded independently on n (and h). In fact,

$$\|A^{-1}\|_2 = \rho(A^{-1}) = \frac{1}{\min_{\lambda \in \sigma(A)} |\lambda|}$$

and the minimum eigenvalue is $a^2 + \pi^2 + \mathcal{O}(h^2)$ such that

$$\|A^{-1}\|_2 \to \frac{1}{a^2 + \pi^2}, \quad h \to 0.$$

(See also note of the course chapter 3.1.1). Then there is C constant independent on n such that $\|A^{-1}\|_2 \leq C$, and we get

$$\|e\|_2 \leq \|A^{-1}\|_2 \|\tau\|_2 \leq C \|\tau\|_2 = \mathcal{O}(h^2).$$

Which proves convergence.
Points: right LTE 6, bound of A^{-1} in 2-norm and bound of the error 8, invertibility of A independently of n 10.

b) The solution of the considered boundary value problem is

$$y(x) = \frac{\cosh(ax)}{\cosh(a)}. \tag{2}$$

Use the identity

$$\cosh(x+h) + \cosh(x-h) = 2 \cosh(x) \cosh(h)$$

to verify that the solution of this difference approximation is

$$Y_j = \frac{\cosh(\theta x_j)}{\cosh(\theta)}, \tag{3}$$

where

$$\theta = (1/h) \cosh^{-1}(1 + \frac{1}{2} a^2 h^2). \tag{4}$$

Solution
From the hypothesis $Y_j = \frac{\cosh(\theta x_j)}{\cosh(\theta)}$, and the equation for the method

$$-\frac{Y_{j-1} - 2Y_j + Y_{j+1}}{h^2} + a^2 Y_j = 0,$$

we see that we need to prove that

$$\frac{-\cosh(\theta x_{j-1}) + 2\cosh(\theta x_j) - \cosh(\theta x_{j+1}) + h^2 a^2 \cosh(\theta x_j)}{\cosh(\theta)} = 0. \tag{5}$$

Now

$$\theta x_{j-1} = \theta x_j - \theta h, \quad \theta x_{j+1} = \theta x_j + \theta h,$$

and using the given identity with x replaced by θx_j and h replaced by θh we get that the left hand side of (5) is

$$\frac{-2\cosh(\theta x_j)\cosh(\theta h) + 2\cosh(\theta x_j) + h^2 a^2 \cosh(\theta x_j)}{\cosh(\theta)}$$

collecting

$$2\frac{\cosh(\theta x_j)}{\cosh(\theta)},$$

we are left with the factor

$$(-\cosh(\theta h) + 1 + \frac{a^2 h^2}{2})$$

which is zero since $\theta = (1/h)\cosh^{-1}(1 + \frac{1}{2}a^2 h^2)$.
Points: right 10, light computation error 8, big computational error 6.

c) Using (3) and (2), by expanding in Taylor series, show directly (without using the local truncation error) that

$$e_j := Y_j - y(x_j) = \frac{1}{24}\, h^2 a^3 \,\frac{(\cosh(ax)\sinh(a) - x\sinh(ax)\cosh(a))}{(\cosh a)^2} + \mathcal{O}(h^4).$$

Hint: From (4), using also Taylor expansion of $\cosh(\theta h)$, show first that

$$\theta^2 = a^2 - \frac{1}{12}\theta^4 h^2 + \mathcal{O}(h^4).$$

From this deduce that

$$\theta = a + \Delta a, \quad \Delta a = -\frac{1}{24}a^3 h^2 + \mathcal{O}(h^4).$$

Finally expand $Y_j = Y_j(a + \Delta a)$ in Taylor series.

Solution
Expanding $\cosh(x)$ in Taylor series around 0 we get

$$\cosh(x) = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \mathcal{O}(x^6),$$

From

$$\theta = (1/h)\cosh^{-1}(1 + \frac{1}{2}a^2h^2)$$

we get

$$\cosh(\theta h) = 1 + \frac{1}{2}a^2h^2$$

and so

$$1 + \frac{1}{2}\theta^2h^2 + \frac{1}{24}\theta^4h^4 + \mathcal{O}(h^6) = 1 + \frac{1}{2}a^2h^2$$

leading to

$$\theta^2 = a^2 - \frac{1}{12}\theta^4h^2 + \mathcal{O}(h^4).$$

Taking square roots and expanding in Taylor series gives

$$\theta = a - \frac{1}{24}\frac{\theta^4}{a}h^2 + \mathcal{O}(h^4),$$

which with

$$\theta^4 = a^4 + \mathcal{O}(h^2),$$

gives finally

$$\theta = a - \frac{1}{24}a^3h^2 + \mathcal{O}(h^4).$$

The final part of the exercise is about expanding in Taylor series $Y_j = Y_j(a+\Delta a)$, this gives

$$Y_j(\theta) = Y_j(a + \Delta a) = f(a + \Delta a) = f(a) + f'(a)\Delta a + \mathcal{O}(\Delta a^2)$$

where

$$f(y) = \frac{\cosh(y\,x_j)}{\cosh(y)}, \quad f'(y) = \frac{\sinh(yx_j)\cosh(y)x_j - \cosh(yx_j)\sinh(y)}{\cosh(y)^2}$$

and so

$$Y_j = \frac{\cosh(a\,x_j)}{\cosh(a)} - \frac{1}{24}a^3h^2\,\frac{\sinh(ax_j)\cosh(a)x_j - \cosh(ax_j)\sinh(a)}{\cosh(a)^2} + \mathcal{O}(h^4),$$

which, using $y(x_j) = \frac{\cosh(a\,x_j)}{\cosh(a)}$, gives the desired result.
Points: First or second part 4, last part assuming the two first parts 6, first and second part 6, first or second part and last part 8, all parts 10.

Appendix

- $$\cosh(x) := \frac{e^x + e^{-x}}{2}, \quad \sinh(x) := \frac{e^x - e^{-x}}{2}.$$

- $$\frac{y(x_{j-1}) - 2y(x_j) + y(x_{j+1})}{h^2} = y''(x_j) + \frac{h^2}{12} y^{IV}(x_j) + \mathcal{O}(h^4).$$

Learning outcome:

Knowledge	**L1**	Understanding of error analysis of difference methods: consistency, stability, convergence of difference schemes.
	L2	Understanding of the basics of the finite element method.
Skills	**L3**	Ability to choose and implement a suitable discretization scheme given a particular PDE, and to design numerical tests in order to verify the correctness of the code and the order of the method.
	L4	Ability to analyze the chosen discretization scheme, at least for simple PDE-test problems.
	L5	Ability to attack the numerical linear algebra challenges arising in the numerical solution of PDEs.
General competence	**L6**	Ability to present in oral and written form the numerical and analytical results obtained in the project work.
	L7	Ability to apply acquired mathematical knowledge in linear algebra and calculus to achieve the other goals of the course.