NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for
TMA4212 Numerical solution of differential equations with difference methods

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Examination time (from–to): :00- :00

Permitted examination support material: C: Approved simple pocket calculator is allowed. The tex book by Strikwerda, the book by Süli and Mayers, and the official note of the TMA4212 course are allowed. Photocopies on 2D finite elements (4 pages) are allowed. Old exams with solutions are not allowed.

Language: English

Number of pages: 4

Number pages enclosed: 0

Checked by:

Date Signature

The learning outcome has been published on the course webpage since the start of the semester and on the official description of the course. We have identified seven goals **L1** to **L7** that should be achieved, see appendix. Learning outcome **L6**, **L3** and to some extent **L4** and **L5** have been tested through the project work. We here test further the achievement of **L4** and **L5** as well as of the remaining goals.

Problem 1 (**L2, L5, L7**) Consider the two-point boundary value problem

$$-u'' + u = f(x), \quad x \in (0,1), \quad u(0) = 0, \quad u(1) = 0,$$

with $f \in C^2[0,1]$.

a) Find the weak formulation of this problem. State the Galerkin method for this problem on a uniform subdivision

$$0 = x_0 < x_1 < \cdots < x_n = 1, \qquad n \geq 2,$$

with $h = x_i - x_{i-1}$. Use piecewise linear finite element basis functions.

b) Write down explicitly the linear system of equations that needs to be solved to compute the numerical solution to the problem. Show that the Galerkin method has only one solution.

c) Analyse the properties and sparsity of the matrix of the linear system one has to solve. Propose suitable numerical methods for the solution of this linear system. Justify your answers.

d) Assume $n = 3$, $f = 1$, find u^h.

Problem 2 (**L1, L4, L7**)

Consider the trianlge Ω bounded by the lines $y = 0$, $y = 2 + 2x$, and $y = 2 - 2x$ in the (x, y)-plane. Consider the numerical solution of Poisson's equation

$$u_{xx} + u_{yy} + f(x, y) = 0$$

on Ω, with Dirichlet conditions given at all points on the boundary, using a grid of size $\Delta x = \Delta y = \frac{1}{N}$, and using variable step-size near the boundary.

a) The goal of this exercise is to obtain a bound for the truncation error of the type

$$\max_{i,j} |\tau_{i,j}| \leq K\,\Delta x^q,$$

for a suitable constant K and a suitable integer q, and for the indexes (i,j) ranging over all the grid points. The solution of the considered Poisson equation is assumed to be sufficiently smooth and to have bounded partial derivatives on Ω.

Find the truncation error $\tau_{i,j}$ of the standard five point difference scheme at internal points, and at points adjacent to the boundary.

b) Show how to choose the constant C so that a maximum principle may be applied to the mesh function

$$u(x_i, y_j) - U_{i,j} + Cy_j^2.$$

Deduce that the error $u(x_i, y_j) - U_{i,j}$ is at least first order in the mesh size.

c) The necessity for a special scheme near the boundary could be avoided by using a rectangular mesh with $\Delta y = 2\Delta x$. Find the linear system you need to solve to implement such method with $\Delta x = 0.5$ and $\Delta y = 1$.

Problem 3 (**L1, L4 L7**)

a) Consider the difference scheme

$$U_i^{n+1} = U_i^n + \frac{k}{h^2}\left(U_{i-1}^n - 2U_i^n + U_{i+1}^n\right) - k\,U_i^n,$$

for the numerical approximation of $u_t = u_{xx} - u$ with homogeneous Dirichlet boundary conditions and with initial function $u(x,0) = f(x)$, on the space interval $[0,1]$, and with $t \geq 0$.

The local truncation error for this method is of second order in h and first order in k.

Prove Lax-Richtmyer stability of the scheme under the condition $\frac{k}{h^2} \leq \frac{1}{2}$, and deduce convergence.

b) Perform a Von-Neumann stability analysis for this scheme.

Problem 4 (L1, L4, L7)

The characteristics of the equation

$$u_t + au_x = 0, \quad 0 \leq x \leq 1$$

when $a = a(t) = 2t$ are

$$x(t) = x_0 + t^2.$$

Consider the following method applied to the given equation

$$\frac{U_i^{n+1} - U_i^n}{k} + 2nk\,\frac{U_i^n - U_{i-1}^n}{h} = 0.$$

Use the characteristics to find the CFL condition for this scheme.

Learning outcome:

Knowledge	**L1**	Understanding of error analysis of difference methods: consistency, stability, convergence of difference schemes.
	L2	Understanding of the basics of the finite element method.
Skills	**L3**	Ability to choose and implement a suitable discretization scheme given a particular PDE, and to design numerical tests in order to verify the correctness of the code and the order of the method.
	L4	Ability to analyze the chosen discretization scheme, at least for simple PDE-test problems.
	L5	Ability to attack the numerical linear algebra challenges arising in the numerical solution of PDEs.
General competence	**L6**	Ability to present in oral and written form the numerical and analytical results obtained in the project work.
	L7	Ability to apply acquired mathematical knowledge in linear algebra and calculus to achieve the other goals of the course.