NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for
TMA4212 Numerical solution of differential equations with difference methods

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Examination time (from–to): 15:00-19:00

Permitted examination support material: C: Approved simple pocket calculator is allowed. The text book by *Strikwerda*, the book by *Süli and Mayers*, and the official note of the TMA4212 course (98 pages) are allowed. Photo copies on 2D finite elements (4 pages) are allowed. Rottman is allowed. Old exams with solutions are *not* allowed.

Language: English

Number of pages: 3

Number pages enclosed: 2

Checked by:

Date Signature

The learning outcome has been published on the course webpage and on the official description of the course. The seven learning goals **L1** to **L7** are reported in the appendix. Learning outcome **L6**, **L3** and to some extent **L4** and **L5** have been tested through the project work. We here test further the achievement of **L4** and **L5** as well as **L1, L2, L7**.

Problem 1 (**L2, L7**)
Given $f \in L^2(0,1)$, state the weak formulation of each of the following boundary value problems on the interval $(0,1)$:

a)

$$-u'' + u = f(x), \quad u(0) = 0, \quad u(1) = 0;$$

b)

$$-u'' + u = f(x), \quad u(0) = 0, \quad u'(1) = 1;$$

c)

$$-u'' + u = f(x), \quad u(0) = 0, \quad u(1) + u'(1) = 2.$$

Problem 2 (**L1, L4, L5, L7**)
Construct explicitly the system of linear equations obtained from approximating Poisson's equation

$$u_{xx} + u_{yy} + f(x,y) = 0$$

in the region defined by $x \geq 0$, $y \geq 0$, $x^2 + y \leq 1$. The boundary conditions are $u(x,0) = p(x)$, $u(0,y) = q(y)$, and $u(x, 1-x^2) = r(x)$, where p, q, r and f are given functions.

Use a grid of size $\Delta x = \frac{1}{3}$ and $\Delta y = \frac{1}{2}$. Use the five point formula. Use variable step-size near the right boundary. (Show that you can apply the method correctly. You do not need to rearrange the terms in the form $AU = b$.)

Problem 3 Consider a linear system $Ax = b$ with the following structure:

$$A = \begin{bmatrix} B_m & \tilde{I}_m & O & \dots & O \\ \tilde{I}_m^T & B_{m-1} & \tilde{I}_{m-1} & \ddots & \vdots \\ O & \tilde{I}_{m-1}^T & \ddots & \ddots & O \\ \vdots & \ddots & & \ddots & \tilde{I}_2 \\ O & \dots & O & \tilde{I}_2^T & B_1 \end{bmatrix}$$

where B_k is $k \times k$ tridiagonal with $\alpha > 4$ on the diagonal, $-\tilde{I}_k$ is $k \times (k-1)$ and it is obtained by removing the last column from the $k \times k$ identity matrix i.e.

$$B_k = \begin{bmatrix} \alpha & -1 & 0 & \dots & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & & & & \\ \vdots & & & \ddots & -1 \\ 0 & \dots & 0 & -1 & \alpha \end{bmatrix}, \qquad \tilde{I}_k = -\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \end{bmatrix}.$$

Is A is symmetric and positive definite? Explain your answer. Can you use the Conjugate Gradient method to approximate the solution of this linear system iteratively?

Problem 4 (**L1, L4, L7**)

a) Consider the equation

$$u_t + a\, u_x = 0, \quad 0 \le x \le 1, \quad t \ge 0,$$

when $a = a(x) = x - \frac{1}{2}$, with a given initial function $u(x, 0)$ for $0 \le x \le 1$. Show that the characteristics are

$$x(t) = e^t\,(x_0 - \frac{1}{2}) + \frac{1}{2}.$$

Explain why you do not need to impose any boundary conditions.

b) The upwind method applied to the equation is

$$\begin{aligned} \tfrac{U_i^{n+1}-U_i^n}{k} + \left(x_i - \tfrac{1}{2}\right) \tfrac{U_i^n - U_{i-1}^n}{h} &= 0, \quad x_i > \tfrac{1}{2} \\ \tfrac{U_i^{n+1}-U_i^n}{k} + \left(x_i - \tfrac{1}{2}\right) \tfrac{U_{i+1}^n - U_i^n}{h} &= 0, \quad x_i < \tfrac{1}{2}, \end{aligned}$$

and we are considering a uniform mesh $x_i = i\,h$, $i = 0, \dots, N$, $h = \frac{1}{N}$, and with k the temporal step-size.

Consider the initial function $u(x, 0) = |x - \frac{1}{2}|$, $N = 4$ and $k = h$. Compute the error of the method in $x = 1/4$ and $x = 3/4$ and at time $t = k$. Recall that $u(x, t) = u(e^{-t}(x - \frac{1}{2}) + \frac{1}{2}, 0)$.

Problem 5 (**L1, L4, L7**)
Consider the boundary value problem

$$-y'' + a^2 y = 0, \quad y(-1) = 1, \qquad y(1) = 1.$$

Consider the following difference approximation to the given problem

$$-\frac{Y_{j-1} - 2Y_j + Y_{j+1}}{h^2} + a^2 Y_j = 0, \qquad j = 1, 2, \ldots, n-1, \quad Y_0 = Y_n = 1, \qquad (1)$$

on equidistant nodes with $h = 2/n$.

a) Find the local truncation error for this method. Then prove convergence in the 2-norm.

b) The solution of the considered boundary value problem is

$$y(x) = \frac{\cosh(ax)}{\cosh(a)}. \qquad (2)$$

Use the identity

$$\cosh(x+h) + \cosh(x-h) = 2\,\cosh(x)\,\cosh(h)$$

to verify that the solution of the difference approximation (1) is

$$Y_j = \frac{\cosh(\theta x_j)}{\cosh(\theta)}, \qquad (3)$$

where

$$\theta = (1/h)\cosh^{-1}(1 + \frac{1}{2}a^2h^2). \qquad (4)$$

c) Using (3) and (2), by expanding in Taylor series, show directly (without using the local truncation error) that

$$e_j := Y_j - y(x_j) = \frac{1}{24}\,h^2 a^3\,\frac{(\cosh(ax)\sinh(a) - x\sinh(ax)\cosh(a))}{(\cosh a)^2} + \mathcal{O}(h^4).$$

Hint: From (4), using also Taylor expansion of $\cosh(\theta h)$, show first that

$$\theta^2 = a^2 - \frac{1}{12}\theta^4 h^2 + \mathcal{O}(h^4).$$

From this deduce that

$$\theta = a + \Delta a, \quad \Delta a = -\frac{1}{24}a^3h^2 + \mathcal{O}(h^4).$$

Finally expand $Y_j = Y_j(a + \Delta a)$ in Taylor series.

Appendix

- $$\cosh(x) := \frac{e^x + e^{-x}}{2}, \quad \sinh(x) := \frac{e^x - e^{-x}}{2}.$$
- $$\frac{y(x_{j-1}) - 2y(x_j) + y(x_{j+1})}{h^2} = y''(x_j) + \frac{h^2}{12} y^{IV}(x_j) + \mathcal{O}(h^4).$$

Learning outcome:

Knowledge	**L1**	Understanding of error analysis of difference methods: consistency, stability, convergence of difference schemes.
	L2	Understanding of the basics of the finite element method.
Skills	**L3**	Ability to choose and implement a suitable discretization scheme given a particular PDE, and to design numerical tests in order to verify the correctness of the code and the order of the method.
	L4	Ability to analyze the chosen discretization scheme, at least for simple PDE-test problems.
	L5	Ability to attack the numerical linear algebra challenges arising in the numerical solution of PDEs.
General competence	**L6**	Ability to present in oral and written form the numerical and analytical results obtained in the project work.
	L7	Ability to apply acquired mathematical knowledge in linear algebra and calculus to achieve the other goals of the course.