NTNU – Trondheim
Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for
TMA4212 Numerisk løsning av differensialligninger med differansemetoder

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Problem 1

a) The grid has $h = 0.1$ and $k = 0.002$. We use central differences in the space direction and forward Euler in time. The method is

$$U_m^{n+1} = U_m^n + \frac{k}{h^2}\delta_x^2 U_m^n.$$

The Dirichlet conditions $u(0,t) = 1$ gives $U_0^n = 1$ for all n. The initial condition $u(x,0) = 1$ gives $U_m^0 = 1$ for $m = 0, \dots, M$ and $M = \frac{1}{M} = 10$. The Robin boundary conditions (4) are discretized using central differences:

$$\frac{U_{M+1}^n - U_{M-1}^n}{2h} = U_M^n, \quad U_{M+1}^n = U_{M-1}^n + 0.2\,U_M^n$$

From the discretization adopted above for nodes inside the domain we get

$$U_M^{n+1} = 0.2\,U_{M-1}^n + 0.6\,U_M^n + 0.2\,U_{M+1}^n,$$

and inserting the obtained expression for U_{M+1}^n we get

$$U_M^{n+1} = 0.2\,U_{M-1}^n + 0.6\,U_M^n + 0.2\,U_{M-1}^n + 0.04\,U_M^n$$

$$U_M^{n+1} = 0.4\,U_{M-1}^n + 0.64\,U_M^n,$$

at the right hand side. So in summary

$$\begin{aligned}
U_1^{n+1} &= 0.2 + 0.6\,U_1^n + 0.2U_2^n \quad m = 1 \\
U_m^{n+1} &= 0.2U_{m-1}^n + 0.6\,U_m^n + 0.2U_{m+1}^n \quad m = 2, \dots, M-1 \\
U_M^{n+1} &= 0.4\,U_{M-1}^n + 0.64U_M^n \quad m = M.
\end{aligned}$$

b) We have $U_{M-2}^0 = U_{M-1}^0 = U_M^0 = 1$. This gives

$$U_{M-1}^1 = 0.2\,U_{M-2}^0 + 0.6U_{M-1}^0 + 0.2U_M^0 = 1,$$

$$U_M^1 = 0.4U_{M-1}^0 + 0.64U_M^0 = 1.04$$

and

$$U_M^2 = 0.4U_{M-1}^1 + 0.64U_M^1 = 0.4 \cdot 1 + 0.64 \cdot 1.04 = 1.0656.$$

So the numerical solution in $(1, 0.002)$ is 1.04, and in $(1, 0.004)$ is 1.0656.

c) The boundary condition (4) is in this case discretized with

$$\frac{U_M^n - U_{M-1}^n}{h} = U_M^n, \quad U_M^n = U_{M-1}^n + 0.1\, U_M^n$$

and

$$U_M^n = \frac{10}{9} U_{M-1}^n.$$

So in summary

$$\begin{aligned} U_1^{n+1} &= 0.2 + 0.6\, U_1^n + 0.2 U_2^n \quad m = 1 \\ U_m^{n+1} &= 0.2 U_{m-1}^n + 0.6\, U_m^n + 0.2 U_{m+1}^n \quad m = 2, \ldots, M-1 \\ U_M^{n+1} &= \frac{10}{9} U_{M-1}^{n+1} \quad m = M. \end{aligned}$$

d) With the last method the numerical solution in $(1, 0.002)$ is $\frac{10}{9}$, and in $(1, 0.004)$ is $\frac{92}{81}$.

e) Here one should discuss the conditional stability for Euler's method.

Problem 2

a) Considering $U = (U_1, \ldots, U_8)$ and using the boundary conditions and the method proposed in the exercise one gets a linear system

$$AU = F$$

with matrix

$$A = \begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 \\ -2 & 0 & 0 & 4 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 4 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 4 \end{bmatrix}$$

b) By using the divergence theorem and the assumed Neumann boundary conditions (over the whole boundary) we get that one cannot have a solution unless $f = 0$. Assuming $f = 0$ one gets that all constants are solutions to the equation, i.e the solution is not unique. The problem either has no solution or infinitely many solutions. So the problem is not well posed.

Problem 3 Recall the definition of domain of dependence of a numerical method (see the course note). The equation for the characteristics is

$$\frac{dx(t)}{dt} = a,$$

which gives that the characteristic through (x^*, t^*) is

$$x - x^* = a(t - t^*).$$

The CFL condition is a necessary condition for convergence and states that the characteristics should never leave the domain of dependence of the numerical method. In other words the CFL condition guarantees the availability of necessary data upon which the solution of the PDE problem at the point (x^*, t^*) builds. Without this information there is no hope to construct a convergent numerical approximation in (x^*, t^*).

If $\alpha_{-1} = 0$ $\alpha_0 \neq 0$ $\alpha_1 \neq 0$ then the domain of dependence is intersecting the x-axis in the interval $[x^*, x^* + \frac{t^*}{p}]$. We assume here that $t^* > 0$. So, since the characteristics for this problem are straight lines the CFL is satisfied if

$$x^* \leq x^* - at^* \leq x^* + \frac{t^*}{p}$$

and

$$0 \leq -at^* \leq \frac{t^*}{p},$$

$$-\frac{1}{p} \leq a \leq 0.$$

So we must require

$$a \leq 0, \quad \text{and} \quad |a|p \leq 1.$$

But if $\alpha_{-1} \neq 0$ $\alpha_0 \neq 0$ $\alpha_1 = 0$ then the domain of dependence is intersecting the x-axis in the interval $[x^* - \frac{t^*}{p}, x^*]$. And the CFL is satisfied if

$$x^* - \frac{t^*}{p} \leq x^* - at^* \leq x^*$$

and

$$0 \leq a \leq \frac{1}{p}.$$

So we must require

$$a \geq 0, \quad \text{and} \quad |a|p \leq 1.$$

Problem 4

a) Multiplying by a test function $v \in X$ where

$$X = \{w \,|\, w \in H^1(\Omega) \,\text{and}\, w = 0 \,on\, \partial_D\Omega\}.$$

Integrating by parts we obtain the weak formulation

$$\int_\Omega \nabla u \cdot \nabla v \, dA = \int_\Omega fv \, dA,$$

and we note that the boundary terms vanish: on $\partial_D\Omega$ because v is zero, and on $\partial_N\Omega$ because $\partial_{\mathbf{n}}u$ is zero. Then

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v \, dA$$

and

$$l(v) := \int_\Omega fv \, dA,$$

and we can rewrite the above as

$$a(u, v) = l(v).$$

b) Let

$$X_H = \{w \in X \,|\, w(x, y) = ax + by + c\},$$

and so w is zero on the nodes $2, 3, 4, 5, 6, 7, 8$ (see figure) then

$$X_H = \text{span}\{\phi_1\},$$

where ϕ_1 is the pyramid function at the node 1. The Galerkin method is:
Find $U \in X_H$ such that

$$a(U, V) = l(V), \quad \forall\, V \in X_H.$$

The numerical solution is therefore

$$U = U_1\phi_1$$

and we rewrite the Galerkin problem as
Find $U_1 \in \mathbb{R}$ such that

$$U_1 a(\phi_1, \phi_1) = l(\phi_1).$$

c) We get

$$a(\phi_1, \phi_1) = \int_\Omega \nabla\phi_1 \cdot \nabla\phi_1 \, dA = \int_{K_1 \cup K_3 \cup K_6} 2 \, dA = 6A.$$

Where K_i is the element number i in the figure (triangle i) and $A = \frac{1}{2}$ is the area if one triangle. So

$$a(\phi_1, \phi_1) = 3.$$

For the right hand side we get

$$l(\phi_1) = \int_\Omega f\phi_1 \, dA = f3 \int_{K_1} \phi_1 \, dA,$$

and we have used the fact that f is constant.

We get

$$\int_{K_1} \phi_1 \, dA = \int_0^1 \int_0^{(1-x)} (1 - x - y) \, dx \, dy = \frac{1}{6}.$$

So $l(\phi_1) = \frac{1}{2} f$.

d) We can now solve the the Galerkin problem
Find $U_1 \in \mathbb{R}$ *such that*

$$U_1 \, a(\phi_1, \phi_1) = l(\phi_1).$$

We get

$$U_1 = \frac{l(\phi_1)}{a(\phi_1, \phi_1)} = \frac{1}{6} f,$$

and the finite element numerical solution is

$$u_h = \frac{1}{6} f \phi_1.$$