NTNU – Trondheim
Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for
TMA4212 Numerical solution of differential equations with difference methods

Academic contact during examination: Elena Celledoni

Phone: 7359 3541

Examination date: xx. august 2014

Examination time (from–to): xx:00-xx:00

Permitted examination support material: C: Approved simple pocket calculator is allowed. The tex book by Strikwerda, the book by Süli and Mayers, and the official note of the TMA4212 course (98 pages) are allowed. Photocopies on 2D finite elements (4 pages) are allowed. Old exams with solutions are not allowed.

Language: English

Number of pages: 9

Number pages enclosed: 2

Checked by:

Date Signature

The learning outcome has been published on the course webpage since the start of the semester and on the official description of the course. We have identified seven goals **L1** to **L7** that should be achieved, see appendix. Learning outcome **L6**, **L3** and to some extent **L4** and **L5** have been tested through the project work. We here test further the achievement of **L4** and **L5** as well as of the remaining goals.

Problem 1 (**L2, L5, L7**) Consider the two-point boundary value problem

$$-u'' + u = f(x), \quad x \in (0,1), \quad u(0) = 0, \quad u(1) = 0,$$

with $f \in C^2[0,1]$.

a) Find the weak formulation of this problem. State the Galerkin method for this problem on a uniform subdivision

$$0 = x_0 < x_1 < \cdots < x_n = 1, \qquad n \geq 2,$$

with $h = x_i - x_{i-1}$. Use piecewise linear finite element basis functions.

Solution
The weak form for this problem is:

Find $u \in H_0^1(0,1)$ such that

$$A(u,v) = \ell(v), \qquad \forall v \in H_0^1(0,1),$$

where

$$A(u,v) := \int_0^1 (u'v' + uv)\,dx, \qquad \ell(v) := \int_0^1 fv\,dx.$$

Taken the approximation space

$$X_h := \{w^h \in H_0^1(0,1) \,|\, w^h = \sum_{i=1}^{n-1} w_i\,\varphi_i, \quad w_i \in \mathbf{R}, \quad i = 1,\dots,n-1\}$$

with φ_i, $i = 1,\dots,n-1$, are the hat functions (piecewise linear functions on $(0,1)$ and such that $\varphi_i(x_j) = \delta_{j,i}$), the Galerkin method is

Find $u^h \in X_h$ such that

$$A(u^h, v^h) = \ell(v^h), \qquad \forall v^h \in X_h.$$

b) Write down explicitly the linear system of equations that needs to be solved to compute the numerical solution to the problem. Show that the Galerkin method as only one solution.

Solution
We observe that $A(u, v) = A(v, u)$ and A is linear in both arguments (because of the linearity of integrals. For this reason assuming

$$u^h := \sum_{i=1}^{n-1} u_i \, \varphi_i,$$

we have

$$A(u^h, v^h) = \ell(v_h), \quad \forall v^h \in X_h \Leftrightarrow A(u^h, \varphi_j) = \ell(\varphi_j), \quad j = 1, \dots, n-1.$$

From which we get

$$\sum_{j=1}^{n-1} u_j A(\varphi_j, \varphi_i) = \ell(\varphi_i), \qquad i = 1, \dots, n-1.$$

Collecting the coefficients $u_1, \dots, u_{n-1}$ into a vector $U := [u_1, \dots, u_{n-1}]^T$ and denoting by $M_{i,j} := A(\varphi_i, \varphi_j)$ the entries of a $(n-1) \times (n-1)$ matrix, we get the linear system

$$MU = f,$$

$f := [\ell(\varphi_1), \dots, \ell(\varphi_{n-1})]^T$. More precisely we get

$$A(\varphi_i, \varphi_{i+1}) = \int_{x_i}^{x_{i+1}} (\varphi_i' \, \varphi_{i+1}' + \varphi_i \, \varphi_{i+1}) \, dx = -\frac{1}{h} + \frac{h}{6}, \tag{1}$$

$$A(\varphi_i, \varphi_i) = \int_{x_{i-1}}^{x_{i+1}} (\varphi_i' \, \varphi_i' + \varphi_i \, \varphi_i) \, dx = \frac{2}{h} + \frac{2h}{3}, \tag{2}$$

the symmetry of A implies $A(\varphi_i, \varphi_{i+1}) = A(\varphi_{i+1}, \varphi_i)$. Since the hat functions and their derivatives are compactly supported we get $A(\varphi_i, \varphi_j) = A(\varphi_j, \varphi_i) = 0$ for $j \geq i + 2$, meaning that the matrix is tridiagonal.

Because of the symmetry of A, M is symmetric. What is important here is that M is invertible. For this it is sufficient to prove that M is positive definite:

$$v^T M v = \sum_{i,j} v_i A(\varphi_j, \varphi_i) v_j = A(v^h, v^h) = \int_0^1 ((v_h')^2 + (v^h)^2) \, dx = \|v^h\|_{H_0^1}^2,$$

and $\| \cdot \|_{H_0^1}$ is an Hilbert space norm, so $\|v^h\|_{H_0^1}^2 = 0$ if and only if v^h is zero (i.e. $v_1, \dots, v_{n-1}$ are all zero). As a consequence M is positive definite, therefore all its eigenvalues are positive and its determinant is different form zero. The linear system (i.e. the Galerkin method) has a unique solution.

c) Analyse the properties and sparsity of the matrix of the linear system one has to solve. Propose suitable numerical methods for the solution of this linear system. Justify your answers.

Solution
As we have seen M is symmetric and positive definite. By the properties of the hat functions we have also seen that the matrix is tridiagonal and by (1) and (2) it is also diagonally dominant. Therefore we can use a Thomas algorithm for this system (we do not need pivoting because the matrix is diagonally dominant). Alternatively we can use a Conjugate Gradient method to solve the linear system, for example using the diagonal D of M as a preconditioner. The CG method is well defined as the matrix is symmetric and positive definite. Using such preconditioner would amount at applying the CG mehtod with the inner product

$$\langle \cdot, \cdot \rangle_D = \langle \cdot, D \cdot \rangle$$

with $\langle \cdot, \cdot \rangle$ denoting the Euclidean inner product.

d) Assume $n = 3$, $f = 1$, find u^h.

Solution
We obtain a 2×2 linear system

$$\begin{bmatrix} 6+\frac{2}{9} & -3+\frac{1}{18} \\ -3+\frac{1}{18} & 6+\frac{2}{9} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

We get $u_1 = u_2 = 0.1017$ and $u^h = 0.1017(\varphi_1 + \varphi_2)$.

Problem 2 (L1, L4, L7)

To obtain a numerical solution of Poisson's equation

$$u_{xx} + u_{yy} + f(x, y) = 0$$

on the trianlge bounded by the lines $y = 0$, $y = 2+2x$, and $y = 2-2x$ with Dirichlet conditions given at all points on the boundary, use a grid of size $\Delta x = \Delta y = \frac{1}{N}$. Use variable step-size near the boundary.

a) Find the leading error terms in the truncation error of the standard five point difference scheme at internal points, and at points adjacent to the boundary.

Solution
Figure 1 is a picture of the domain and grid with $h = \frac{1}{4}$. On the odd horizontal gridlines (j odd) we need to use a step of size $h/2$ instead of h to get hold of the boundary value. On the even horizontal gridlines (j even) the nodes of the grid coincide with boundary nodes. The vertical lines intersect the boundary always on grid nodes. So for all j odd, the first and last node (from left to right) on row j of the grid have an adjacent node on the boundary (respectively to the left and right side) at only $h/2$ distance instead of h. For these nodes we need to use variable step size in the x-direction. The general discretization of the Laplacian for this problem becomes

$$\left(\frac{U_{i+1,j}-U_{i,j}}{\Delta x_{i+1}}-\frac{U_{i,j}-U_{i-1,j}}{\Delta x_i}\right)\frac{2}{\Delta x_{i+1}+\Delta x_i}+\left(\frac{U_{i,j+1}-2U_{i,j}+U_{i,j-1}}{h^2}\right)=-f(x_i,y_j),$$

where

$$\Delta x_i := x_i - x_{i-1},$$

$\Delta x_i = h/2$ for nodes near the boundary and $\Delta x_i = h$ otherwise.

For all nodes with j even and nodes with j odd but not adjacent to the boundary the local truncation error is the same as for the usual five point formula for equidistant nodes (see note of the course chapter 6) and the leading error terms depend on the fourth partial derivatives of u with respect to x and y:

$$\tau_{i,j} = \frac{h^2}{12}(u_{xxxx}(x_i,y_j) + u_{yyyy}(x_i,y_j)) + \mathcal{O}(h^4).$$

For nodes with j odd adjacent to the boundary we can distingush between left and right boundary and get different formulae but similar results. We consider therefore here only the case of a node close to the left boundary, the discretization becomes

$$\left(\frac{U_{i+1,j}-U_{i,j}}{h}-\frac{U_{i,j}-U_{i-1,j}}{h/2}\right)\frac{4}{3h}+\left(\frac{U_{i,j+1}-2U_{i,j}+U_{i,j-1}}{h^2}\right)=-f(x_i,y_j),$$

replacing $U_{i,j}$ with $u_{i,j} = u(x_i, y_j)$ and using Taylor expansion in the first term at the left hand side we get

$$\left(u_x(x_i,y_j)+\frac{h}{2}u_{xx}(x_i,y_j)+\frac{h^2}{3!}u_{xxx}(x_i,y_j)-u_x(x_i,y_j)+\frac{h}{4}u_{xx}(x_i,y_j)-\frac{h^2}{8\,3!}u_{xxx}(x_i,y_j)+\mathcal{O}(h^3)\right)\frac{4}{3h}$$

giving

$$u_{xx}(x_i,y_j)+\frac{7h}{36}u_{xxx}(x_i,y_j)+\mathcal{O}(h^2)$$

so

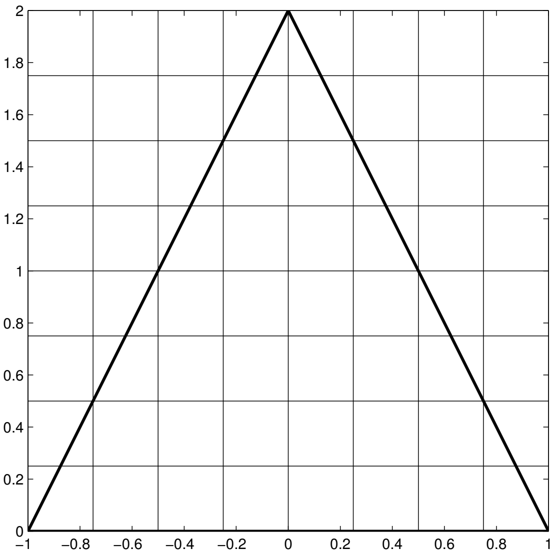
$$\tau_{i,j} = u_{xx}(x_i, y_j) + \frac{7h}{36} u_{xxx}(x_i, y_j) + \mathcal{O}(h^2) + u_y y(x_i, y_j) + \frac{h^2}{12} u_{yyyy}(x_i, y_j) + \mathcal{O}(h^4) + f(x_i, y_j)$$

and finally using the Poisson equation in the point (x_i, y_j) one gets

$$\tau_{i,j} = \frac{7h}{36} u_{xxx}(x_i, y_j) + \mathcal{O}(h^2)$$

a first order error with leading error constant depending on the third derivative in x.

Figure 1: Domain and grid.



b) Show how to choose the constant C so that a maximum principle may be applied to the mesh function

$$u(x_i, y_j) - U_{i,j} + Cy_j^2.$$

Deduce that the error in the solution is at least first order in the mesh size.

Solution

$$C = \max_P \{ \left| \partial_x^4 u \Big|_P \right|, \quad \left| \partial_y^4 u \Big|_P \right|, \quad \left| \partial_x^3 u \Big|_P \right| \}$$

where the maximum is taken over all grid points P. Then the obtained grid function

$$V_{i,j} := u(x_i, y_j) - U_{i,j} + Cy_j^2$$

can be used to prove convergence of order 1 in h by the discrete maximum principle (see last section of chapter 6 in the course note).

c) The necessity for a special scheme near the boundary could be avoided by using a rectangular mesh with $\Delta y = 2\Delta x$. Find the linear system you need to solve to implement such method with $\Delta x = 0.5$ and $\Delta y = 1$.

Solution
In this case we have just one unknown corresponding to the point with coordinates $P = (0, 1)$, which is the only internal node of the grid. The linear system collapses to a single equation:

$$(U_w - 2U_p + U_e)4 + (U_n - 2U_p + U_s) = -F_p,$$

leading to the following expression for U_p by means of known boundary functions and F,

$$U_p = \frac{F_p + 4U_w + 4U_e + U_n + U_s}{10}.$$

Problem 3 (L1, L4 L7)

a) Consider the difference scheme

$$U_i^{n+1} \quad = \quad U_i^n + \frac{k}{h^2}\left(U_{i-1}^n - 2U_i^n + U_{i+1}^n\right) - k\,U_i^n,$$

for the numerical approximation of $u_t = u_{xx} - u$ with homogeneous Dirichlet boundary conditions and with initial function $u(x, 0) = f(x)$, on the space interval $[0, 1]$, and with $t \geq 0$.

The local truncation error for this method is of second order in h and first order in k.

Prove convergence of the scheme under the condition $\frac{k}{h^2} \leq \frac{1}{2}$.

Solution
For the matrix-stability analysis for this problem we write the method in the form

$$U^{n+1} = U^n + \frac{k}{h^2}AU^n - kU^n,$$

where A is the tridiagonal $M \times M$ matrix with -2 on the main diagonal and 1 on first the super- and sub-diagonal. The method is then

$$U^{n+1} = CU^n,$$

with

$$C = (I - kI + \frac{k}{h^2}A) = V(I - kI + \frac{k}{h^2}\Lambda)V^T.$$

C is a symmetric matrix and therefore a necessary and sufficient condition for stability is that there exist ν independent on h and k such that

$$\rho(C) \leq 1 + \nu k,$$

where $\rho(C)$ denotes the spectral radius of C. The eigenvalues of C are

$$1 - k + \frac{k}{h^2}\lambda_m, \quad \lambda_m = -4\sin^2\phi_m, \quad \phi_m = \frac{m\pi}{2(M+1)}, \ m = 1, \ldots, M.$$

And

$$\rho(C) = \max_m |1 - k + \frac{k}{h^2}\lambda_m|.$$

Since $\lambda_m < 0$

$$1 - k + \frac{k}{h^2}\lambda_m \leq 1$$

for all m. Let $r = \frac{k}{h^2}$. To obtain

$$\rho(C) \leq 1 + k,$$

we require

$$-1 - k \leq 1 - k + r\lambda_m = 1 - k - 4r\sin^2\frac{m\pi}{2(M+1)}.$$

This condition is the same as

$$2 \geq 4r\sin^2\frac{m\pi}{2(M+1)} = 4r\sin^2(\frac{\pi}{2} - h\frac{\pi}{2}) = 4r\cos^2(h\frac{\pi}{2}).$$

Leading to the condition

$$r \leq \frac{1}{2\cos^2(\frac{\pi}{2}h)} = \frac{1}{2} + \frac{1}{8}\pi^2h^2 + \mathcal{O}(h^4),$$

so a more restrictive and sufficient stability condition is

$$r \leq \frac{1}{2}.$$

By the Lax equivalence theorem a consistent difference scheme applied to a linear PDE is convergent if and only if it is stable. Since we have proved stability and consistency is assumed in the exercise text, the method must be convergent.

b) Perform a Von-Neumann stability analysis for this scheme.

Solution
Inserting as usual $U_m^n = \xi^n e^{i\beta x_m}$ we get

$$\xi = (1-k) + r(e^{-i\beta h} - 2 + e^{i\beta h})$$

and

$$\xi = (1-k) - 4r\sin^2(\frac{\beta h}{2}).$$

The condition

$$|\xi| \le 1 + \nu k,$$

with ν independent on h and k should be satisfied to have Von-Neumann stability. We try with $\nu = 1$. Obviously

$$1 - k - 4r\sin^2(\frac{\beta h}{2}) \le 1 + k,$$

while

$$-1 - k \le 1 - k - 4r\sin^2(\frac{\beta h}{2})$$

gives the stability condition

$$r \le \frac{1}{2\sin^2(\frac{\beta h}{2})}.$$

Which is analogous to the matrix-stability condition of question **a**).

Problem 4 (L1, L4, L7)

The characteristics of the equation

$$u_t + a u_x = 0, \quad 0 \le x \le 1$$

when $a = a(t) = 2t$ are

$$x(t) = x_0 + t^2.$$

Consider the following method applied to the equation is

$$\frac{U_i^{n+1} - U_i^n}{k} + 2nk\,\frac{U_i^n - U_{i-1}^n}{h} = 0.$$

Use the characteristics to find the CFL condition for this scheme.

Solution We observe that the characteristic curves intersect the x-axis in $x_0 = x(t) - t^2$. In particular the characteristic through a point (x^*, t^*) in the (x, t)-plane satisfies

$$x^* = x(t^*), \quad x_0 = x^* - (t^*)^2, \quad x(t) = x^* - (t^*)^2 + t^2.$$

Writing the characteristic as a function of x, x^* and t^* we get

$$t = \sqrt{x - x^* + (t^*)^2},$$

and are monotone increasing functions of x. Differentiating $t(x)$ with respect to x we get

$$\frac{dt}{dx} = \frac{1}{2} \frac{1}{\sqrt{x - x^* + (t^*)^2}},$$

a monotone decreasing function of x. So the tangent in x^* to this function remains above the function for all $x \leq x^*$. In $x = x^*$ we have

$$\frac{1}{2} \frac{1}{\sqrt{x^* - x^* + (t^*)^2}} = \frac{1}{2\,t^*}.$$

We rewrite the method in the form

$$U_i^{n+1} = U_i^n - \frac{2nk^2}{h}(U_i^n - U_{i-1}^n)$$

and find that for a point (x^*, t^*) in the (x, t)-plane, its domain of dependence is

$$(x^* - t^*/p, x^*), \qquad p = \frac{k}{h}.$$

So on the right side the domain of dependence is delimited by the vertical line $x = x^*$, while on the left side by the line through (x^*, t^*) with slope $p = \frac{k}{h}$. In order for the characteristics to be contained in the domain of dependence of the method, it is sufficient that the slope of this line is smaller than or equal to the slope of the characteristic in x^*, i.e. the CFL condition is

$$p \leq \frac{1}{2\,t^*}.$$

Learning outcome:

Knowledge	**L1**	Understanding of error analysis of difference methods: consistency, stability, convergence of difference schemes.
	L2	Understanding of the basics of the finite element method.
Skills	**L3**	Ability to choose and implement a suitable discretization scheme given a particular PDE, and to design numerical tests in order to verify the correctness of the code and the order of the method.
	L4	Ability to analyze the chosen discretization scheme, at least for simple PDE-test problems.
	L5	Ability to attack the numerical linear algebra challenges arising in the numerical solution of PDEs.
General competence	**L6**	Ability to present in oral and written form the numerical and analytical results obtained in the project work.
	L7	Ability to apply acquired mathematical knowledge in linear algebra and calculus to achieve the other goals of the course.