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Contact during the exam:
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EXAM in TMA4212

Monday 12th August 2013
Time: 9:00–13:00

Allowed aids: Approved simple pocket calculator. All written and handwritten material from the course.

Learning outcome **L6** and **L3** have been tested through the project work[1].

Problem 1 *Learning outcome* **L2, L7**

Given that α is a nonnegative real number, consider the differential equation

$$-u'' + u = f(x), \quad \text{for } x \in (0,1),$$

subject to the boundary conditions

$$u(0) = 0, \qquad \alpha u(1) + u'(1) = 0.$$

a) State the weak formulation of the problem.

b) Using continuous piecewise linear basis functions on a uniform subdivision of $[0,1]$ into elements of size $h = \frac{1}{n}$, $n \geq 2$, write down the finite element approximation to this problem and show that this has a unique solution. Expand u^h in terms of the standard piecewise linear finite element basis functions (hat functions) φ_i, $i = 1, 2, \ldots, n$, by writing

$$u^h(x) = \sum_{i=1}^{n} U_i \varphi_i(x)$$

to obtain a system of linear equations for the vector of unknowns $(U_1, \ldots, U_n)^T$.

[1]The learning outcome is published on the course webpage and on the official description of the course.

c) Suppose $\alpha = 0$, $f(x) \equiv 1$ and $h = \frac{1}{3}$. Solve the resulting system of linear equations and compute the corresponding numerical solution.

d) Compare the numerical solution with the exact solution u by finding the values of the error function $u_h - u$ in the nodes of the finite element discretization. You can find u assuming $u(x) = c_1 e^{-x} + c_2 e^{x} + 1$, and determining c_1 and c_2 imposing the boundary conditions.

Problem 2 *Learning outcome* **L1, L4, L5, L7**
Consider the linear PDE

$$u_t + u_{xxx} = 0, \quad x \in [0, 1], \quad t \geq 0,$$

with periodic boundary conditions. Consider the grid $x_m = hm$, $h = 1/M$, $m = 1, \ldots, M$. Discretize with central finite differences in space and the Forward Euler method in time (forward differences in time).

Use the following central differences approximation of the third derivative

$$u_{xxx}\big|_{(x_m,t)} = \frac{u(x_{m+3}, t) - 3u(x_{m+1}, t) + 3u(x_{m-1}, t) - u(x_{m-3}, t)}{8h^3} + \mathcal{O}(h^2).$$

a) Perform a Von Neumann stability analysis of the method.

b) Consider now a Crank-Nicolson method obtained applying central differences in space (as in point (a)) and the trapezoidal rule in time. Perform a stability analysis of the method following the techniques explained in chapter 5 of the note (see page 55 chapter 5.6 of the note). Find under which restrictions on h and k the method is stable.

Hint: You might use the fact that the chosen discretization of the third derivative operator is a skew-symmetric matrix (and circulant). See also exercise (c) below.

c) Consider now the method obtained applying the Backward Euler method in time (backward differences in time). The linear system to be solved at each time-step is of the form

$$(I - kA)\, x = b,$$

with A $M \times M$ circulant and skew-symmetric. (See the appendix on circulant matrices). Consider the Jacobi iterative method to solve this linear system. Assume M and h are fixed and find under which conditions on k the Jacobi iteration is guaranteed to converge.

Appendix

A circulant matrix is a matrix of the type

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \dots & c_{M-2} & c_{M-1} \\ c_{M-1} & c_0 & c_1 & c_2 & \ddots & c_{M-2} \\ c_{M-2} & c_{M-1} & c_0 & c_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & c_1 \\ c_1 & \dots & \dots & \dots & c_{M-1} & c_0 \end{bmatrix},$$

and the first row in the matrix is

$$\mathbf{c}^T = [c_0, c_1, c_2, \dots, c_{M-2}, c_{M-1}],$$

and it is determining the whole matrix. Circulant matrices can be diagonalized using the Fourier matrix Ω, where

$$\Omega_{k,l} = \frac{1}{\sqrt{M}} \exp\left(2\pi i \cdot (k-1)(l-1)/M\right), \quad i = \sqrt{-1}$$

and $C = \Omega^H \Lambda \Omega$ with Λ a diagonal matrix, and $\Omega^H \Omega = I$. The eigenvalues of C, i.e. the diagonal elements of Λ, are given by

$$\sqrt{M} \cdot \Omega^H \mathbf{c}.$$