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EXAM in TMA4212

Thursday 23rd May 2013

Allowed aids: Approved simple pocket calculator. All written and handwritten material from the course. Learning outcome **L6** and **L3** have been tested through the project work[1].

Problem 1 *Learning outcome* **L2, L7**
Consider the differential equation

$$-(p(x)u')' + r(x)u = f(x), \quad \text{for } x \in (a, b),$$

with $p \in C^1[a, b]$, $p(x) \geq c_0 > 0$ for all $x \in [a, b]$, $r \in C^0[a, b]$, $r(x) \geq 0$ for all $x \in [a, b]$ and $f \in L_2[a, b]$, subject to the boundary conditions

$$-p(a)u'(a) + \alpha\, u(a) = A, \qquad p(b)u'(b) + \beta\, u(b) = B,$$

where α and β are positive real numbers and A and B are real numbers.

a) Show that the weak formulation of the boundary value problem is

$$\text{find} \quad u \in H^1(a, b), \text{ such that } \mathcal{A}(u, v) = \ell(v) \text{ for all } v \in H^1(a, b),$$

where

$$\mathcal{A}(u, v) = \int_a^b [p(x)u'(x)v'(x) + r(x)u(x)v(x)]\, \mathrm{d}x + \alpha u(a)v(a) + \beta u(b)v(b),$$

and

$$\ell(v) = \langle f, v \rangle + Av(a) + Bv(b).$$

[1]The learning outcome is published on the course webpage and on the official description of the course.

b) We will now consider a finite element approximation of the boundary value problem based on this weak formulation using piecewise linear finite element basis functions on the subdivision

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

on the interval $[a, b]$ and with constant step size. Show that the finite element method gives rise to a set of $n+1$ simultaneous linear equations with $n+1$ unknowns $u_i = u^h(x_i)$, $i = 0, 1, \ldots, n$. Show that this linear system has a unique solution. Comment on the structure of the matrix $M \in \mathbf{R}^{(n+1)\times(n+1)}$ of the linear system: Is M symmetric? Is M positive definite? Is M tridiagonal? Explain your answers.

Problem 2 *Learning outcome* **L5, L7**

Consider Laplace equation in 2D on a square domain $\Omega = [0, 1] \times [0, 1]$,

$$u_{xx} + u_{yy} = 0,$$

with Dirichlet boundary conditions,

$$u = g, \quad \text{on } \partial\Omega.$$

Use the 5-points formula to discretize the equation on the grid $h = \frac{1}{M+1}$, $x_i = i\,h$, $i = 1, \ldots, M$, and $y_j = j\,h$, $j = 1, \ldots, M$ and obtain a linear system of algebraic equations

$$A\,\mathbf{u} = \mathbf{b},$$

where $\mathbf{u}$ is a vector whose components are the numerical approximations of u on the grid of the discretization $u_{i,j} \approx u(x_i, y_j)$, ordered as follows:

$$\mathbf{u} := [u_{1,1}, u_{2,1}, \ldots, u_{M,1}, u_{1,2}, \ldots, u_{M,2}, \ldots, u_{1,M}, \ldots, u_{M,M}]^T.$$

Consider the use of the conjugate gradient method to solve this linear system.

a) Write out A explicitly.

Recall then the following formula for the eigenvalues of A:

$$\mu_{j,l} = \frac{2}{h^2}\left(\cos(\pi j h) - 1\right) + \frac{2}{h^2}\left(\cos(\pi l h) - 1\right), \quad j = 1, \ldots, M,\; l = 1, \ldots, M.$$

Show that $-A$ is a positive definite matrix whose eigenvalues lie in an interval $[a, b]$ with $\sqrt{\frac{a}{b}}$ approximately $\frac{\pi h}{2}$.

b) Assume you are given an arbitrary initial guess $\mathbf{u}_0$. Use the theorem for the convergence of the conjugate gradient method to estimate the number of iterations K necessary to obtain a relative error below the tolerance ε, that is

$$\frac{\|e_K\|_A}{\|e_0\|_A} \leq \varepsilon.$$

Recall that for a vector $\mathbf{v}$,

$$\|\mathbf{v}\|_A := \sqrt{\mathbf{v}^T A \mathbf{v}}.$$

Find an estimate for K assuming $M+1 = 100$ and $\varepsilon = 10^{-7}$.

Problem 3 *Learning outcome* **L1, L4, L7**
Consider the linear PDE

$$u_t + u_{xxx} = 0, \quad u(-\infty, t) = u(\infty, t) = 0, \quad x \in \mathbf{R}, \quad t \geq 0.$$

Consider the interval $[-L, L]$ with $L > 0$ sufficiently large, consider the grid $x_m = -L + hm$, $h = 2L/M$, $m = 0, \ldots, M$. Discretise with central finite differences in space and the Backward Euler method in time (backward differences in time), let $u(x_0, t) = u(x_M, t) = 0$.

Use the following central differences approximation of the third derivative

$$u_{xxx}\big|_{(x_m,t)} = \frac{u(x_{m+3}, t) - 3u(x_{m+1}, t) + 3u(x_{m-1}, t) - u(x_{m-3}, t)}{8h^3} + \mathcal{O}(h^2).$$

Show that the obtained method is Von Neumann stable.

Problem 4 *Learning outcome* **L1, L4, L7**
Let r and α be positive real numbers. Show that if $[w_1, \ldots, w_M]^T$ satisfies

$$-rw_{m-1} + (1 + 2r + \alpha)w_m - rw_{m+1} = v_m, \quad 1 \leq m \leq M-1, \quad w_0 = w_M = 0,$$

then

$$\max_{0 \leq m \leq M} |w_m| \leq \max_{1 \leq m \leq M-1} |v_m|.$$

Use this to show that Backward Euler converges for arbitrary $r = k/h^2$ on the problem

$$u_t = u_{xx} - u, \quad u(x, 0) = f(x), \quad u(0, t) = g_0(t), \quad u(1, t) = g_1(t).$$

Hint: You will have to take $w_m = e_m^{n+1}$ where $e_m^n := U_m^n - u(x_m, t_m)$ is the error, and U_m^n is the numerical approximation given by the Backward Euler method.