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# EXAM in TMA4212

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Time: 9:00–13:00

Allowed aids: Approved simple pocket calculator. All written and handwritten material form the course.

## Problem 1

**a)** Explain briefly the CFL stability condition. Use the equation $u_t + au_x = 0$, and the scheme

$$u_m^{n+1} = \alpha_{-1}\, u_{m-1}^n + \alpha_0\, u_m^n + \alpha_1\, u_{m+1}^n$$

to explain the theory. What happens when $\alpha_{-1} = 0$ and $a > 0$ ?

**Solution**. The CFL condition is a necessary condition for convergence and it says that to get convergence of the numerical approximation to the solution at the point $(x^*, t^*)$ in the $x$-$t$ plane, the characteristic of $u_t + au_x = 0$ through $(x^*, t^*)$ must not leave the domain of dependence of the numerical scheme and in particular must intersect the $x$-axis within the domain of dependence of the numerical scheme.

For the presented scheme, provided $\alpha_{-1}$ and $\alpha_1$ are both non zero for all time levels, the domain of dependence is $[x^* - t^*/p, x^* + t^*/p]$, and the CFL conditions amounts at requiring that

$$|a\, p| \leq 1.$$

Conversely, if $\alpha_{-1} = 0$ for all time levels, the domain of dependence of the numerical scheme is $[x^*, x^* + t^*/p]$ while the characteristic of $u_t + au_x = 0$ through $(x^*, t^*)$ intersects the $x$-axis in $x_0 = x^* - at^*$ which is not in the domain of dependence. This implies that the CFL condition is violated.

**b)** Consider now $u_t + au_x = 0$ and show that the scheme

$$\frac{\frac{1}{2}\left(u_{m+1}^{n+1}+u_{m-1}^{n+1}\right)-u_m^n}{k}+a\frac{u_{m+1}^{n+1}-u_{m-1}^{n+1}}{2h}=0$$

is von Neumann stable if $|a\,p|$ is greater than or equal to 1, $p = k/h$.
**Solution**. We set as usual $u_m^n = \xi^n\, e^{m\beta h}$ and substitute in the method and get

$$\xi\,\cos\beta h - 1 + ap\xi\, i\,\sin\beta h = 0,$$

$$\xi\big(\cos\beta h + ap\, i\,\sin\beta h\big) = 1,$$

$$\xi = \big(\cos\beta h + ap\, i\,\sin\beta h\big)^{-1} = \frac{\cos\beta h - ap\, i\,\sin\beta h}{\cos^2\beta h + a^2p^2\,\sin^2\beta h},$$

and

$$|\xi|^2 = \big(\cos^2\beta h + a^2p^2\,\sin^2\beta h\big)^{-1}.$$

Then

$$\cos^2\beta h + a^2p^2\,\sin^2\beta h \geq 1,$$

$$a^2p^2 + (1-a^2p^2)\cos^2\beta h \geq 1,$$

$$(a^2p^2-1)(1-\cos^2\beta h) \geq 0,$$

$$|ap| \geq 1.$$

**Problem 2** Given $\alpha$ a nonnegative real number, consider the differential equation

$$-u'' + u = f(x), \quad x \in (0,1)$$

subject to the boundary conditions

$$u(0) = 0, \quad \alpha\, u(1) + u'(1) = 0, \alpha \geq 0.$$

**a)** State the weak formulation of the problem.
**Solution**. We multiply both sides of the equation by a test function $v \in H^1(0,1)$ vanishing on the left boundary of the domain, and integrate between 0 and 1. Using integration by parts and $v(0) = 0$, we obtain

$$-\,u'v|_0^1 + \int_0^1 u'v'\,dx + \int_0^1 uv\,dx = \int_0^1 fv\,dx, \quad \forall v \in H^1(0,1), \quad v(0) = 0$$

which is the weak formulation of the problem. By using the boundary condition we get

$$\alpha u(1)v(1) + \int_0^1 u'v'\,dx + \int_0^1 uv\,dx = \int_0^1 fv\,dx, \quad \forall v \in H^1(0,1), \quad v(0) = 0$$

setting

$$a(u,v) := \alpha u(1)v(1) + \int_0^1 u'v'\,dx + \int_0^1 uv\,dx, \quad \langle f, v\rangle := \int_0^1 f\,v\,dx$$

we can write equivalently

$$a(u,v) = \langle f, v\rangle,$$

(Galerkin formulation).

**b)** Using continuous piecewise linear basis functions on a uniform subdivision of $[0,1]$ into elements of size $h = 1/n$, $n \geq 2$, write down the finite element approximation to this problem and show that this has a unique solution $u^h$. Expand $u^h$ in terms of the standard piecewise linear finite element basis functions (hat functions) $\varphi_i$, $i = 1, 2, \ldots, n$, by writing

$$u^h(x) = \sum_{i=1}^n U_i \varphi_i(x)$$

to obtain a system of linear equations for the vector of unknowns $(U_1, \ldots, U_n)^T$.
**Solution**. We follow the suggested ansatz for $u^h$

$$u^h(x) = \sum_{i=1}^n U_i \varphi_i(x)$$

and the boundary conditions

$$u^h(0) = 0, \quad \alpha\, u^h(1) + {u^h}'(1) = 0.$$

Here the piecewise linear basis functions are

$$\varphi_j(x) = \begin{cases} \frac{(x - x_{j-1})}{h} & x_{j-1} \leq x < x_j, \\ \frac{(x_{j+1} - x)}{h} & x_j \leq x \leq x_{j+1}, \\ 0 & \text{otherwise}, \end{cases} \qquad j = 1, \ldots, n-1,$$

$$\varphi_n(x) = \begin{cases} \frac{(x - x_{n-1})}{h} & x_{n-1} \leq x \leq x_n, \\ 0 & \text{otherwise}, \end{cases}$$

so each of these basis functions vanishes in 0 and so does $u^h$ (the first boundary condition is automatically satisfied). The derivatives of these basis functions are

$$\varphi_j'(x) = \begin{cases} \frac{1}{h} & x_{j-1} \leq x < x_j, \\ -\frac{1}{h} & x_j \leq x \leq x_{j+1}, \\ 0 & \text{otherwise}, \end{cases} \qquad j = 1, \ldots, n-1,$$

$$\varphi_n'(x) = \begin{cases} \frac{1}{h} & x_{n-1} \leq x < x_n, \\ 0 & x = x_n, \\ 0 & \text{otherwise.} \end{cases}$$

We now write the linear system for the unknowns $(U_1, \dots, U_n)^T$ which we determine by stating the Galerkin method:

$$a(u^h, \varphi_i) = \langle f, \varphi_i \rangle, \quad i = 1, \dots, n,$$

and

$$\sum_{j=1}^{n} a(\varphi_j, \varphi_i) U_j = \langle f, \varphi_i \rangle, \quad i = 1, \dots, n,$$

giving us the linear system

$$M\mathbf{u} = \mathbf{b}, \quad \mathbf{u} := (U_1, \dots, U_n)^T,$$

and $\mathbf{b} := (\langle f, \varphi_1 \rangle, \dots, \langle f, \varphi_n \rangle)$. The matrix $M$ has entries

$$M_{i,j} = a(\varphi_j, \varphi_i), \quad i, j = 1, \dots, n-1,$$

is symmetric and positive definite because $a(\cdot, \cdot)$ is symmetric and $a(v, v) = \int_0^1 (v'^2 + v^2)\, dx + \alpha v(1)^2 > 0$ for $v \neq 0$, implying that for a vector $\mathbf{v} \neq 0$ $\mathbf{v} \in \mathbf{R^n}$ one has $\mathbf{v}^T M \mathbf{v} = a(v, v) > 0$, where $v = \sum_{i=1}^n v_i \varphi_i$ and $v_i$ the $i$-th component of $\mathbf{v}$. Therefore since $M$ is positive definite (it has also all eigenvalues different from zero) it is invertible and the linear system $M\mathbf{u} = \mathbf{b}$ has a unique solution.

To find the entries of $M$ and $\mathbf{b}$ we need to compute the integrals

$$a(\varphi_j, \varphi_i) = \alpha \varphi_j(1) \varphi_i(1) + \int_0^1 \varphi_j' \varphi_i'\, dx + \int_0^1 \varphi_i \varphi_j\, dx, \quad \langle f, \varphi_i \rangle = \int_0^1 f\, \varphi_i\, dx.$$

**c)** Suppose that $\alpha = 0$, $f(x) \equiv 1$ and $h = 1/3$. Solve the resulting system of linear equations and give an expression for $u^h$.
**Solution**.

The matrix $M$ is in this case a $3 \times 3$ tridiagonal and symmetric matrix (as observed previously). It can be easily checked that $M_{1,3} = a(\varphi_3, \varphi_1) = 0$ and by symmetry we get $M_{3,1} = 0$. To find $M$ we need to compute the three diagonal entries

$$M_{i,i} = \int_0^1 \varphi_i' \varphi_i'\, dx + \int_0^1 \varphi_i \varphi_i\, dx, \quad i = 1, 2, 3,$$

and the two entries

$$M_{1,2} = \int_0^1 \varphi_2' \varphi_1'\, dx + \int_0^1 \varphi_2 \varphi_1\, dx = M_{2,1},$$

$$M_{2,3} = \int_0^1 \varphi_3' \varphi_2' \, dx + \int_0^1 \varphi_3 \varphi_2 \, dx = M_{3,1}.$$

We get

$$M_{1,1} = M_{2,2} = \frac{2}{h} + \frac{2}{3}h, \quad M_{3,3} = \frac{1}{h} + \frac{1}{3}h,$$

$$M_{1,2} = M_{2,3} = -\frac{1}{h} + \frac{1}{6}h.$$

Further we need to find

$$b_1 = \int_0^1 \varphi_1 \, dx = b_2 = \int_0^1 \varphi_2 \, dx = h, \quad b_3 = \int_0^1 \varphi_3 \, dx = \frac{1}{2}h.$$

We get

$$U_1 = \frac{1}{2}\frac{h(4a^2 - d^2 - 2ad)}{a(-3d^2 + 4a^2)}, U_2 = \frac{2h(-d+a)}{-3d^2 + 4a^2}, U_3 = \frac{1}{2}\frac{h(d^2 - 4ad + 4a^2)}{a(-3d^2 + 4a^2)}$$

with $a = \frac{1}{h} + \frac{h}{3}, \quad d = -\frac{1}{h} + \frac{1}{6}h.$

**Problem 3** Consider the partial differential equation

$$u_t = i\, u_{xx}, \quad x \in (0,1)$$

with $i = \sqrt{-1}$, and periodic boundary conditions. Consider the finite difference scheme

$$u_m^{n+1} = u_m^n + i\,\frac{k}{2h^2}\left(\delta_x^2 u_m^n + \delta_x^2 u_m^{n+1}\right),$$

and $\frac{1}{h^2}\delta_x^2 u(x,t)$ is the central difference approximation of the second derivative of $u(x,t)$ with respect to $x$. Prove Lax-Richtmyer stability of the scheme. Under which circumstances is the scheme convergent?

**Solution**. The scheme can be written as

$$U^{n+1} = U^n + i\,\frac{k}{2h^2}A(U^n + U^{n+1})$$

with $U^n = (U_1^n, \ldots, U_M^n)^T$ and $U_0^n = U_M^n$, and $A$ the usual central difference matrix discretization of the Laplacian. Then the scheme can be then written as

$$U^{n+1} = CU^n, \quad C = (I - i\,rA)^{-1}(I + i\,rA),$$

with $r = k/2h^2$. The matrix $C$ is symmetric and has eigenvalues

$$(1 - ir\lambda_j)^{-1}(1 + ir\lambda_j), \quad j = 1, \ldots, M$$

where $\lambda_j$ are the eigenvalues of $A$. In the case $C$ is symmetric $\rho(C) \leq 1$ is a sufficient condition for Lax-Richtmyer stability and in this case it is obviously satisfied, because all values

$$\left|(1 - ir\lambda_j)^{-1}(1 + ir\lambda_j)\right| = 1, \quad j = 1, \ldots, M.$$

By Lax equivalence theorem a consistent difference scheme is stable if and only if it is convergent. Since this scheme is consistent it is also convergent.