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Contact during the exam:
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EXAM in TMA4212

13th August 2011
Time: 9:00–13:00

Allowed aids: Approved simple pocket calculator. All written and handwritten material form the course.

Problem 1

a) Explain briefly the CFL condition. Use the equation $u_t + au_x = 0$, and the scheme

$$u_m^{n+1} = \alpha_{-1}\, u_{m-1}^n + \alpha_0\, u_m^n + \alpha_1\, u_{m+1}^n$$

with $\alpha_{-1} \neq 0$, $\alpha_0 \neq 0$ and $\alpha_1 \neq 0$ constant values for all time levels, to explain the theory. What happens when $\alpha_{-1} = 0$ and $a > 0$?

b) Consider now $u_t + au_x = 0$ and show that the scheme

$$\frac{\frac{1}{2}\left(u_{m+1}^{n+1} + u_{m-1}^{n+1}\right) - u_m^n}{k} + a\frac{u_{m+1}^{n+1} - u_{m-1}^{n+1}}{2h} = 0$$

is von Neumann stable if $|a\,p|$ is greater than or equal to 1, here $p = k/h$.

Problem 2 Given α a nonnegative real number, consider the differential equation

$$-u'' + u = f(x), \quad x \in (0,1)$$

subject to the boundary conditions

$$u(0) = 0, \quad \alpha\, u(1) + u'(1) = 0,\ \alpha \geq 0.$$

a) State the weak formulation of the problem.

b) Using continuous piecewise linear basis functions on a uniform subdivision of $[0,1]$ into elements of size $h = 1/n$, $n \geq 2$, write down the finite element approximation to this problem and show that this has a unique solution u^h. Expand u^h in terms of the standard piecewise linear finite element basis functions (hat functions) φ_i, $i = 1, 2, \ldots, n$, by writing

$$u^h(x) = \sum_{i=1}^{n} U_i \varphi_i(x)$$

to obtain a system of linear equations for the vector of unknowns $(U_1, \ldots, U_n)^T$.

c) Suppose that $\alpha = 0$, $f(x) \equiv 1$ and $h = 1/3$. Solve the resulting system of linear equations and give an expression for u^h.

Problem 3 Consider the partial differential equation

$$u_t = i\, u_{xx}, \quad x \in (0,1)$$

with $i = \sqrt{-1}$, and periodic boundary conditions. Consider the finite difference scheme

$$u_m^{n+1} = u_m^n + i\,\frac{k}{2h^2}\left(\delta_x^2 u_m^n + \delta_x^2 u_m^{n+1}\right),$$

where $\frac{1}{h^2}\delta_x^2 u(x,t)$ is the central difference approximation of the second derivative of $u(x,t)$ with respect to x. Prove stability of the scheme[1]. Under which circumstances is the scheme convergent?

[1] Use the definition of stability given in chapter 5 (Stability, consistency and convergence) of the English version of the note of the course TMA4212.

Appendix. The piecewise linear basis functions are

$$\varphi_j(x) = \begin{cases} \frac{(x-x_{j-1})}{h} & x_{j-1} \le x \le x_j, \\ \frac{(x_{j+1}-x)}{h} & x_j \le x \le x_{j+1}, \\ 0 & \text{otherwise}, \end{cases} \qquad j = 1, \dots, n-1,$$

$$\varphi_n(x) = \begin{cases} \frac{(x-x_{n-1})}{h} & x_{n-1} \le x \le x_n, \\ 0 & \text{otherwise}, \end{cases}$$

and $x_j = j\,h$, $j = 0, \dots, n$, $x_0 = 0$, $x_n = 1$.