Norwegian University of Science and Technology
Department of Mathematical Sciences

Contact during the exam:
Elena Celledoni, tlf. 735 93541

EXAM in TMA4212

Monday 23rd May 2011
Time: 9:00–13:00

Allowed aids: Approved simple pocket calculator. All written and handwritten material form the course.

Problem 1 We approximate the solution of the Poisson equation

$$\Delta u = u_{xx} + u_{yy} = f \quad \text{i} \quad \mathcal{R}, \quad u = 0 \quad \text{på } \partial\mathcal{R}$$

using the finite element method. We use trapezoidal elements with a node in each corner and bilinear element basis functions. The trapezium we consider here has the following corners

$$(h,\ 0)\,, \quad \left(\frac{h}{2},\ \frac{1}{2}\sqrt{3}\,h\right), \quad \left(-\frac{h}{2},\ \frac{1}{2}\sqrt{3}\,h\right), \quad (-h,\ 0)$$

for a suitable choice of the origin such that $h > 0$ is a discretization parameter. Three basis functions are known

$$\psi_1(x,y) = -\frac{1}{6}\sqrt{3}\,\frac{1}{h^2}\,(x+h)\,(2y - h\sqrt{3})$$
$$\psi_2(x,y) = \frac{1}{3}\sqrt{3}\,\frac{1}{h^2}\,(2x+h)\,y$$
$$\psi_3(x,y) = \frac{1}{3}\sqrt{3}\,\frac{1}{h^2}\,(h-2x)\,y$$

a) Find the fourth basis function.

b) This element E has an element stiffness matrix $A^E = [\alpha^E_{pq}]_{p,q=1:4}$. Compute one entry in A^E (you choose which one).

Suggestion. With $x = \xi h,\ y = \eta h$ the following integration formula holds true

$$\int_E \left(a\,\xi^2 + b\,\xi\eta + c\,\eta^2 + d\,\xi + e\,\eta + f\right) dA = h^2\left(\frac{5}{32}\sqrt{3}(a+c) + \frac{1}{2}e + \frac{3}{4}\sqrt{3}f\right)$$

for any abitrary choice of the constants a, b, c, d, e, f where $dA = d(h\xi)d(h\eta)$.

Problem 2 We solve the initial value problem

$$u_t = u_{xx}, \qquad -\infty < x < \infty, \quad t > 0$$
$$u(x,0) = f(x), \qquad -\infty < x < \infty.$$

We let $u^n_m = u(x_m, t_n)$ med $x_m = mh$ og $t_n = nk$. The discretization error is $e^n_m = u^n_m - U^n_m$. We want to obtain an expilicit formula so that the discretization error is $\mathcal{O}(h^2 + k^2)$.

a) Using Taylor series for u^{n+1}_m around (x_m, t_n), explain how one can obtain the formula

$$\frac{U^{n+1}_m - U^n_m}{k} = \left(\frac{1}{h^2}\,\delta^2_x + \frac{1}{2}\,\frac{k}{h^4}\,\delta^4_x\right) U^n_m,$$

and find the order (in h and k) for the local truncation error τ^n_m[1].

b) Find the conditions on $r = k/h^2$ guaranteeing that the method in (**a**) satisfies the criterion for von Neumann (require $|\xi| \le 1$).

c) Let us now assume that there is a constant C such that the local truncation error for the method in (**a**) satisfies

$$|\tau^n_m| \le \mu, \qquad \mu = C(h^2 + k^2)$$

Assume also that $r \le 1/2$ and show that the discretization error

$$e^n_m = u^n_m - U^n_m$$

satisfies

$$\max_m |e^{n+1}_m| \le \max_m |e^n_m| + k\mu,\ n \ge 0$$

Find an upperbound for $|e^n_m|$, with $nk \le T$ for a given T.

[1] The definition for the local truncation error used here is the one given in the English translation of the note for the course TMA4212.

Problem 3 Analize dissipation and dispersion for the Lax-Friedrichs method for $u_t + au_x = 0$:

$$U_j^{n+1} = \frac{1}{2}(U_{j-1}^n + U_{j+1}^n) - \frac{ap}{2}(U_{j+1}^n - U_{j-1}^n), \quad p = \frac{k}{h}.$$