NTNU
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Examination paper for **TMA4205 Numerical Linear Algebra**

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Examination date: 30.11.2017

Examination time (from–to): 09:00–13:00

Permitted examination support material: C: Specified, written and handwritten examination support materials are permitted. A specified, simple calculator is permitted. The permitted examination support materials are:

- Y. Saad: Iterative Methods for Sparse Linear Systems. 2nd ed. SIAM, 2003 (book or printout).
- L. N. Trefethen and D. Bau: Numerical Linear Algebra, SIAM, 1997 (book or photocopy).
- G. Golub and C. Van Loan: Matrix Computations. 3rd ed. The Johns Hopkins University Press, 1996 (book or photocopy).
- E. Rønquist: Note on The Poisson problem in $\mathbb{R}^2$: diagonalization methods (printout).
- M. Grasmair: The singular value decomposition (printout).
- Rottmann, Matematisk formelsamling.
- Your own lecture notes from the course (handwritten).

Language: English

Number of pages: 8

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
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Problem 1 Let

$$A = \begin{pmatrix} 5 & -3 \\ -3 & 1 \end{pmatrix}.$$

Perform one step of the QR-method (for computing eigenvalues) with the shift parameter $\mu = 1$.

Idea of solution:

- We first shift the matrix by $\mu\,\mathrm{Id}$ and obtain

  $$A_\mu := \begin{pmatrix} 4 & -4 \\ -4 & 0 \end{pmatrix},$$

  which has the QR-decomposition

  $$A_\mu = QR = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \sqrt{2} \begin{pmatrix} 4 & -2 \\ 0 & -2 \end{pmatrix}.$$

  (The first column of Q is a normalisation of the first column of A_μ, the second column is orthogonal to that, and R can be found by computing $Q^T A_\mu$.) Multiplying R with Q now yields

  $$RQ = \begin{pmatrix} 6 & 2 \\ 2 & -2 \end{pmatrix}.$$

  Adding back the shift $\mu\,\mathrm{Id}$, we obtain the final result

  $$\tilde{A} = \begin{pmatrix} 7 & 2 \\ 2 & -1 \end{pmatrix}.$$

Problem 2 Consider the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Given $b \in \mathbb{R}^4$, one applies the CG method for the solution of the system $Ax = b$. After at most how many steps will the method find the exact solution?

In addition, perform two steps of the CG method for the solution of the system $Ax = b$ with $b = (-2, 2, 2, -2)^T$. Use the initialisation $x_0 = 0$.

Idea of solution:

- Obviously we have 2 different eigenvalues and thus at most two steps.

  For the iteration, we initialise $r_0 = p_0 = b$ and then compute

$$Ap_0 = \begin{pmatrix} -4 \\ 4 \\ 2 \\ -2 \end{pmatrix}$$

and

$$\alpha_0 = \|r_0\|_2^2/(Ap_0, p_0) = 16/24 = 2/3.$$

Thus

$$x_1 = x_0 + \alpha_0 p_0 = \frac{4}{3}\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}.$$

Now

$$r_1 = r_0 - \alpha_0 A p_0 = \begin{pmatrix} -2 \\ 2 \\ 2 \\ -2 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} -4 \\ 4 \\ 2 \\ -2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 2 \\ -2 \end{pmatrix}.$$

Next,

$$\beta_0 = \|r_1\|^2/\|r_0\|^2 = (16/9)/16 = 1/9,$$

and

$$p_1 = r_1 + \beta_0 p_0 = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 2 \\ -2 \end{pmatrix} + \frac{1}{9}\begin{pmatrix} -2 \\ 2 \\ 2 \\ -2 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 4 \\ -4 \\ 8 \\ -8 \end{pmatrix}.$$

In the second iteration, we obtain

$$Ap_1 = \frac{1}{9}\begin{pmatrix} 8 \\ -8 \\ 8 \\ -8 \end{pmatrix}$$

and

$$\alpha_1 = \|r_1\|^2/(Ap_1, p_1) = (16/9)/(64/27) = 3/4.$$

Thus

$$x_2 = x_1 + \alpha_1 p_1 = \frac{4}{3}\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} + \frac{3}{4}\frac{1}{9}\begin{pmatrix} 4 \\ -4 \\ 8 \\ -8 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \\ -2 \end{pmatrix}.$$

Moreover, we readily obtain that $r_2 = 0$, that is, we have, as expected, found the solution.

Problem 3 We consider the linear system $Ax = b$, where $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite. In order to solve this system, we apply a two-dimensional projection method, where the $(k+1)$-st iterate is computed using the search space and constraint space $\mathcal{K}_k = \mathcal{L}_k = \operatorname{span}\{r^{(k)}, Ar^{(k)}\}$, with $r^{(k)} = b - Ax^{(k)}$ being the current residual.

Using the convergence result for the steepest descent method, show that this method converges for each $x^{(0)} \in \mathbb{R}^n$ to a solution of the linear system.

Idea of solution:

- The iterate $x^{(k+1)}$ is the result of a projection method with A SPD and $\mathcal{K}_k = \mathcal{L}_k$. As a consequence, it solves the optimisation problem

$$\min_{x\in x^{(k)}+\mathcal{K}_k} \|x - x^*\|_A,$$

where $x^* = A^{-1}b$ is the solution of the linear system. Now denote by $x_{\mathrm{SD}}^{(k+1)}$ the result of one step of the steepest descent method. Then by construction $x^{(k+1)} \in x^{(k)} + \operatorname{span}\{r^{(k)}\} \subset x^{(k)} + \mathcal{K}_k$, which implies that

$$\|x^{(k+1)} - x^*\|_A \le \|x_{\mathrm{SD}}^{(k+1)} - x^*\|_A.$$

Next we use that

$$\|x_{\mathrm{SD}}^{(k+1)} - x^*\|_A \le \frac{\lambda_{\max} - \lambda_{\min}}{\lambda_{\max} + \lambda_{\min}} \|x^{(k)} - x^*\|_A.$$

(see Saad, Theorem 5.9). Together, these inequalities imply that

$$\|x^{(k+1)} - x^*\|_A \le \frac{\lambda_{\max} - \lambda_{\min}}{\lambda_{\max} + \lambda_{\min}} \|x^{(k)} - x^*\|_A.$$

Since $\frac{\lambda_{\max}-\lambda_{\min}}{\lambda_{\max}+\lambda_{\min}} < 1$, this proves the convergence of the iterates $x^{(k)}$ to x^*.

Problem 4 Assume that $B \in \mathbb{R}^{n\times n}$ is skew-symmetric, that is, $B^T = -B$. In particular, this implies that the diagonal elements of B are equal to 0. Also, because of the skew-symmetry of B, its eigenvalues are purely imaginary, that is, the eigenvalues of B have the form $\lambda_j = i\mu_j$ with $\mu_1 \geq \mu_2 \geq \ldots \mu_{n-1} \geq \mu_n$. Moreover, $\mu_j = -\mu_{n-j+1}$ for all j.

In the following, we consider the solution of a linear system $Ax = b$, where $A = \operatorname{Id} + B$.

a) Consider the steepest descent method for the solution of the system $Ax = b$. Show that, for this particular case, the step size α chosen by the steepest descent method is in each step equal to 1, and thus this method is identical to the Jacobi method.

Idea of solution:

- The steepest descent method reads

$$x^{(k+1)} = x^{(k)} + \alpha_k(b - Ax^{(k)})$$

with

$$\alpha_k = \frac{\|r^{(k)}\|^2}{(Ar^{(k)}, r^{(k)})}.$$

Because of the skew-symmetry of B we have $(Ar^{(k)}, r^{(k)}) = (r^{(k)} + Br^{(k)}, r^{(k)}) = (r^{(k)}, r^{(k)})$, and thus $\alpha_k = 1$. Thus

$$x^{(k+1)} = (\operatorname{Id} - A)x^{(k)} + b.$$

Since the diagonal elements of A are all equal to 1 (those of B are 0), this is precisely the Jacobi method.

b) Consider now the weighted Jacobi method with weight $0 < \omega \leq 1$ for solving the system $Ax = b$. That is, $x^{(k+1)}$ is defined as

$$x^{(k+1)} := (1 - \omega)x^{(k)} + \omega x_J^{(k+1)},$$

where

$$x_J^{(k+1)} := D^{-1}(E + F)x^{(k)} + D^{-1}b$$

is the result of applying one step of the Jacobi method to the vector $x^{(k)}$.

Assume that $\mu_1 < 1$. Show that the weighted Jacobi method converges for each $0 < \omega \leq 1$.

Idea of solution:

- The weighted Jacobi iteration reads as

$$x^{(k+1)} = (1-\omega)x^{(k)} + \omega(Bx^{(k)} + b) = \Big((1-\omega)\operatorname{Id} + \omega B\Big)x^{(k)} + \omega b.$$

  The iteration matrix of this method is therefore $G_\omega := (1-\omega)\operatorname{Id} + \omega B$, which has the eigenvalues $(1-\omega) - \omega i \mu_j$. Since $|\mu_j| < 1$ for all j, this is (in modulus) strictly smaller than 1, which implies that the iteration converges.

Problem 5 Let $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite, and assume that all diagonal entries of A are equal to 1.

We consider the solution of a linear system $Ax = b$ using a left-preconditioned GMRES method, where the application of a preconditioner M^{-1} to a vector r consists in performing two iterations of the Jacobi method for solving $Az = r$ with initialisation $z^{(0)} = 0$.

a) Write down an algebraic expression for the left-preconditioned matrix $M^{-1}A$ and show that it is symmetric and positive definite provided that $\|A\|_2 < 2$.

Idea of solution:

- Since $D = \operatorname{Id}$ (diagonal part of A), we obtain $M^{-1} = (\operatorname{Id} - A) + \operatorname{Id} = 2\operatorname{Id} - A$. The left preconditioned matrix becomes $(2\operatorname{Id} - A)A$. This is obviously symmetric. For the positive definiteness of this matrix, we observe that its eigenvalues are of the form $2\lambda_j - \lambda_j^2$, where λ_j is an eigenvalue of A. Since A is symmetric and positive definite, its eigenvalues are positive and equal to its singular values, which, because of the assumption $\|A\|_2 < 2$ are all smaller than 2. As $2\lambda - \lambda^2 > 0$ whenever $0 < \lambda < 2$, this proves the positive definiteness of M^{-1}.

b) Assume now that the eigenvalues of the matrix A are all contained in the intervals $(0.1, 0.2)$ and $(1.8, 1.9)$. Estimate, how many iterations of the preconditioned GMRES method are necessary in order to decrease the Euclidean norm of the residual by a factor of 10^{-6}.

Idea of solution:

- The eigenvalues of the preconditioned matrix are of the form $(2-\lambda)\lambda$, where λ is an eigenvalue of A. Thus they are all contained in the interval $(0.19, 0.36)$, which then implies that the condition of the preconditioned matrix is smaller than 36/19. The matrix is SPD, and thus the estimates for the CG method carry over (for the residual instead of the error) to the GMRES method, that is, we have

$$\|Ax^{(k)} - b\|_2 \leq 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k \|Ax^{(0)} - b\|_2.$$

The desired improvement of the accuracy is thus guaranteed in case

$$2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k \leq 10^{-6},$$

which a short computation shows to hold for $k \geq 8$.

Problem 6 Compute the (reduced) SVD of the matrix

$$A = \begin{pmatrix} 22 & 2 & 13 \\ 4 & 14 & 16 \end{pmatrix}.$$

Idea of solution:

- In order to find the SVD $A = U\Sigma V^T$, we compute first the eigenvalues of AA^T. We have

$$AA^T = \begin{pmatrix} 657 & 324 \\ 324 & 468 \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$p(\lambda) = \lambda^2 - 1125\lambda + 202500,$$

which has the zeroes

$$\lambda_\pm = \frac{1125 \pm 675}{2} = \begin{cases} 900 \\ 225 \end{cases}$$

Thus the singular values of A are $\sigma_1 = 30$ and $\sigma_2 = 15$ and

$$\Sigma = \begin{pmatrix} 30 & 0 \\ 0 & 15 \end{pmatrix}.$$

Next we note that $v_1 = (4,3)/5$ is an eigenvector of AA^T for the eigenvalue 900, and thus $v_2 := v_1^\perp = (3,-4)$ is an eigenvector for the eigenvalue 225. As a consequence, we can choose

$$U = \frac{1}{5}\begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}.$$

Finally, we have

$$V = A^T U \Sigma^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 2 \\ 1 & -2 \\ 2 & -1 \end{pmatrix}.$$

Problem 7 The matrix

$$A = \begin{pmatrix} 9.6 & -2.4 & 2.4 & 0 \\ 2.8 & 1.8 & 3.2 & 10 \end{pmatrix}$$

has the reduced singular value decomposition

$$A = U\Sigma V^T$$

with

$$U = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} 12 & 0 \\ 0 & 9 \end{pmatrix} \qquad V^T = \frac{1}{3}\begin{pmatrix} 2 & 0 & 1 & 2 \\ -2 & 1 & 0 & 2 \end{pmatrix}.$$

a) Find the rank one matrix $B \in \mathbb{R}^{2\times 4}$ for which $\|A - B\|_F$ is minimal.

Idea of solution:

- The solution of this optimisation problem is the matrix $U\hat{\Sigma}V^T$, where $\hat{\Sigma}$ is the (rank one) diagonal matrix which has as its only non-zero entry at the top-left corner of the matrix the largest singular value of A. Carrying out the multiplication $U\hat{\Sigma}V^T$ with $\hat{\Sigma} = \begin{pmatrix} 12 & 0 \\ 0 & 0 \end{pmatrix}$, we obtain the solution $B = \begin{pmatrix} 4.8 & 0 & 2.4 & 4.8 \\ 6.4 & 0 & 3.2 & 6.4 \end{pmatrix}$.

b) Use the singular value decomposition of A in order to compute the solution of the problem

$$\min_{x\in\mathbb{R}^4} \|x\|_2^2 \qquad \text{s.t. } Ax = \begin{pmatrix} 0 \\ 5 \end{pmatrix}.$$

Idea of solution:

- The solution of the problem is the vector $x^\dagger = A^\dagger b$ where $b = (0, 5)^T$ is the right hand side of the equation and $A^\dagger = V\Sigma^{-1}U^T$ is the pseudoinverse of A (note that Σ is invertible). In this case we obtain

$$x^\dagger = V\Sigma^{-1}U^T b = \frac{1}{9}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 4 \end{pmatrix}.$$