NTNU
Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4205 Numerical Linear Algebra**

Academic contact during examination: Adrian Kirkeby

Phone: 41 51 70 26

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Permitted examination support material: C: Specified, written and handwritten examination support materials are permitted. A specified, simple calculator is permitted. The permitted examination support materials are:

- Y. Saad: Iterative Methods for Sparse Linear Systems. 2nd ed. SIAM, 2003 (book or printout).
- L. N. Trefethen and D. Bau: Numerical Linear Algebra, SIAM, 1997 (book or photocopy).
- G. Golub and C. Van Loan: Matrix Computations. 3rd ed. The Johns Hopkins University Press, 1996 (book or photocopy).
- E. Rønquist: Note on The Poisson problem in $\mathbb{R}^2$: diagonalization methods (printout).
- M. Grasmair: The singular value decomposition (printout).
- Rottmann, Matematisk formelsamling.
- Your own lecture notes from the course (handwritten).

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Problem 1 Let

$$A = \begin{pmatrix} 5 & -4 \\ -4 & 1 \end{pmatrix}.$$

Perform one step of the QR-method (for computing eigenvalues) with the shift parameter $\mu = 1$.

Problem 2 Consider the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Given $b \in \mathbb{R}^4$, one applies the CG method for the solution of the system $Ax = b$. After at most how many steps will the method find the exact solution?

In addition, perform two steps of the CG method for the solution of the system $Ax = b$ with $b = (-2, 2, 2, -2)^T$. Use the initialisation $x_0 = 0$.

Problem 3 We consider the linear system $Ax = b$, where $A \in \mathbb{R}^{n \times n}$ is symmetric and positive definite. In order to solve this system, we apply a two-dimensional projection method, where the $(k+1)$-st iterate is computed using the search space and constraint space $\mathcal{K}_k = \mathcal{L}_k = \operatorname{span}\{r^{(k)}, Ar^{(k)}\}$, with $r^{(k)} = b - Ax^{(k)}$ being the current residual.

Using the convergence result for the steepest descent method, show that this method converges for each $x^{(0)} \in \mathbb{R}^n$ to a solution of the linear system.

Problem 4 Assume that $B \in \mathbb{R}^{n \times n}$ is skew-symmetric, that is, $B^T = -B$. In particular, this implies that the diagonal elements of B are equal to 0. Also, because of the skew-symmetry of B, its eigenvalues are purely imaginary, that is, the eigenvalues of B have the form $\lambda_j = i\mu_j$ with $\mu_1 \geq \mu_2 \geq \ldots \mu_{n-1} \geq \mu_n$. Moreover, $\mu_j = -\mu_{n-j+1}$ for all j.

In the following, we consider the solution of a linear system $Ax = b$, where $A = \mathrm{Id} + B$.

a) Consider the steepest descent method for the solution of the system $Ax = b$. Show that, for this particular case, the step size α chosen by the steepest descent method is in each step equal to 1, and thus this method is identical to the Jacobi method.

b) Consider now the weighted Jacobi method with weight $0 < \omega \leq 1$ for solving the system $Ax = b$. That is, $x^{(k+1)}$ is defined as

$$x^{(k+1)} := (1-\omega)x^{(k)} + \omega x_J^{(k+1)},$$

where

$$x_J^{(k+1)} := D^{-1}(E+F)x^{(k)} + D^{-1}b$$

is the result of applying one step of the Jacobi method to the vector $x^{(k)}$.

Assume that $\mu_1 < 1$. Show that the weighted Jacobi method converges for each $0 < \omega \leq 1$.

Problem 5 Let $A \in \mathbb{R}^{n \times n}$ be symmetric and positive definite, and assume that all diagonal entries of A are equal to 1.

We consider the solution of a linear system $Ax = b$ using a left-preconditioned GMRES method, where the application of a preconditioner M^{-1} to a vector r consists in performing two iterations of the Jacobi method for solving $Az = r$ with initialisation $z^{(0)} = 0$.

a) Write down an algebraic expression for the left-preconditioned matrix $M^{-1}A$ and show that it is symmetric and positive definite provided that $\|A\|_2 < 2$.

b) Assume now that the eigenvalues of the matrix A are all contained in the intervals $(0.1, 0.2)$ and $(1.8, 1.9)$. Estimate, how many iterations of the preconditioned GMRES method are necessary in order to decrease the Euclidean norm of the residual by a factor of 10^{-6}.

Problem 6 Compute the (reduced) SVD of the matrix

$$A = \begin{pmatrix} 22 & 2 & 13 \\ 4 & 14 & 16 \end{pmatrix}.$$

Problem 7 The matrix

$$A = \begin{pmatrix} 9.6 & -2.4 & 2.4 & 0 \\ 2.8 & 1.8 & 3.2 & 10 \end{pmatrix}$$

has the reduced singular value decomposition

$$A = U\Sigma V^T$$

with

$$U = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} 12 & 0 \\ 0 & 9 \end{pmatrix} \qquad V^T = \frac{1}{3}\begin{pmatrix} 2 & 0 & 1 & 2 \\ -2 & 1 & 0 & 2 \end{pmatrix}.$$

a) Find the rank one matrix $B \in \mathbb{R}^{2\times 4}$ for which $\|A - B\|_F$ is minimal.

b) Use the singular value decomposition of A in order to compute the solution of the problem

$$\min_{x\in\mathbb{R}^4} \|x\|_2^2 \qquad \text{s.t. } Ax = \begin{pmatrix} 0 \\ 5 \end{pmatrix}.$$