1 Compute the (reduced) singular value decomposition and the pseudoinverse of the matrix

$$A = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

Possible solution

The reduced singular value decomposition of A has the form

$$A = U\Sigma V^T$$

with $U \in \mathbb{R}^{1\times 1}$ orthogonal (that is, $U = \pm 1$), $\Sigma \in \mathbb{R}^{1\times 1}$ containing the singular value(s) of A (that is, $\Sigma = (\sigma)$ with σ being the only singular value of A) and $V \in \mathbb{R}^{1\times 3}$. We compute

$$A^T A = \begin{pmatrix} 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 9,$$

which shows that $3 = \sqrt{9}$ is the (single) singular value of A. Setting $\Sigma = (3)$ and $U = (1)$, we obtain $V = A^T/3$ and

$$A = U\Sigma V^T = (1)(3) \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}.$$

2 Assume that $A \in \mathbb{R}^{n\times n}$ is skew-symmetric. Show that the singular values of A are precisely the absolute values of the eigenvalues of A.

Possible solution

The singular values of a matrix A are exactly the square roots of the eigenvalues of the matrix $A^T A$. Since A is skew-symmetric, it follows that $A^T A = -A^2$. Moreover, if λ is an eigenvalue of A, then $-\lambda^2$ is an eigenvalue of $-A^2$. Thus σ is a singular value of A, if and only if $\sigma = \sqrt{-\lambda^2}$ with λ being an eigenvalue of A; since the eigenvalues of A are purely imaginary, we have $\sqrt{-\lambda^2} = |\lambda|$, which proves the claim.

3 Compute the (reduced) singular value decomposition of the matrix

$$A = \begin{pmatrix} 10 & 10 \\ -1 & 7 \\ 5 & 5 \\ -2 & 14 \end{pmatrix}.$$

Additionally, compute the pseudoinverse $A^\dagger$ of A and use it in order to solve the least squares problem

$$\min_{x \in \mathbb{R}^2} \frac{1}{2}\|Ax - b\|_2^2 \qquad \text{where } b = \begin{pmatrix} 7 \\ -5 \\ 1 \\ 1 \end{pmatrix}$$

Possible solution

We have

$$A^T A = \begin{pmatrix} 130 & 90 \\ 90 & 370 \end{pmatrix}$$

with characteristic polynomial

$$p(\lambda) = \lambda^2 - 500\lambda + 40000$$

and (consequently) eigenvalues

$$\lambda_1 = 400 \qquad \text{and} \qquad \lambda_2 = 100.$$

The singular values of A are therefore

$$\sigma_1 = \sqrt{\lambda_1} = 20 \qquad \text{and} \qquad \sigma_2 = \sqrt{\lambda_2} = 10,$$

and we have

$$\Sigma = \begin{pmatrix} 20 & 0 \\ 0 & 10 \end{pmatrix}.$$

Next we compute an eigenbasis of $A^T A$, which provides the matrix V in the singular value decomposition of A. We note that

$$A^T A - 100I = \begin{pmatrix} 30 & 90 \\ 90 & 270 \end{pmatrix}$$

from which we obtain the normalised eigenvector v_2 for the eigenvalue λ_2

$$v_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

Since the eigenvectors of $A^T A$ are orthogonal, we can choose

$$v_1 = v_2^\perp = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Thus

$$V = (v_1, v_2) = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix}.$$

Next we need to compute U. To that end, we note that the equation $A = U\Sigma V^T$ implies that

$$U = AV\Sigma^{-1},$$

that is,

$$U = \frac{1}{\sqrt{10}} \begin{pmatrix} 2 & 2 \\ 1 & -1 \\ 1 & 1 \\ 2 & -2 \end{pmatrix}.$$

Thus the singular value decomposition of A is

$$A = U\Sigma V^T = \frac{1}{\sqrt{10}} \begin{pmatrix} 2 & 2 \\ 1 & -1 \\ 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 20 & 0 \\ 0 & 10 \end{pmatrix} \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix}.$$

Next, we compute the pseudoinverse

$$A^\dagger = V\Sigma^{-1}U^T = \begin{pmatrix} 0.07 & -0.025 & 0.03 & -0.05 \\ 0.01 & 0.025 & 0.005 & 0.05 \end{pmatrix}.$$

Finally, the solution of the optimisation problem

$$\min_{x \in \mathbb{R}^2} \frac{1}{2} \|Ax - b\|_2^2 \qquad \text{where } b = \begin{pmatrix} 7 \\ -5 \\ 1 \\ 1 \end{pmatrix}$$

is simply the vector

$$x^\dagger = A^\dagger b = \begin{pmatrix} 0.6 \\ 0 \end{pmatrix}.$$

4 Compute the pseudoinverse of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Using this particular matrix, show that the pseudoinverse of a matrix does not necessarily satisfy the relation $(A^\dagger)^2 = (A^2)^\dagger$.

Possible solution

We first compute a singular value decomposition $A = U\Sigma V^T$ of A. We have

$$AA^T = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

with eigenvalues 2 and 0 and corresponding eigenvectors $(1,0)$ and $(0,1)$. Thus we have only one non-zero singular value $\sigma_1 = \sqrt{2}$, and we can write

$$A = \sigma_1 u_1 v_1^T = \sqrt{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} v_1^T.$$

Obviously, $v_1^T = (1/\sqrt{2}, 1/\sqrt{2})$, and we have

$$A = \sigma_1 u_1 v_1^T = \sqrt{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1/\sqrt{2}, 1/\sqrt{2}).$$

Now the pseudoinverse computes as

$$A^\dagger = \frac{1}{\sigma_1} v_1 u_1^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} (1,0) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

In particular, we have

$$(A^\dagger)^2 = \frac{1}{4} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Next we note that

$$A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = A,$$

and therefore

$$(A^2)^\dagger = A^\dagger = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Obviously, we have $(A^\dagger)^2 \neq (A^2)^\dagger$.