



- 1 Assume that $A \in \mathbb{R}^{m \times n}$ with $m > n$ has full rank and that $b \in \mathbb{R}^m$. Consider the following iteration (the *CGNR-method*):

Set $r \leftarrow b$, $z \leftarrow A^T x$, $p \leftarrow z$, $s \leftarrow \|z\|_2^2$;

while not yet converged **do**

$w \leftarrow Ap$;

$\alpha \leftarrow s/\|w\|_2^2$;

$x \leftarrow z + \alpha p$;

$r \leftarrow r - \alpha w$;

$z \leftarrow A^T r$;

$s_{\text{old}} \leftarrow s$;

$s \leftarrow \|z\|_2^2$;

$\beta \leftarrow s/s_{\text{old}}$;

$p \leftarrow z + \beta p$;

end

- a) Show that this algorithm converges to a solution of the least squares problem

$$\min_x \|Ax - b\|_2.$$

- b) Denote by x_k the result after k iterations of this algorithm. Show that x_k minimises $\|Ax - b\|_2$ among all vectors $x \in \mathcal{K}_k(A^T A, A^T b)$.

Hint: Show that this algorithm implements the CG-method for the normal equations.

- 2 (Cf. Exercise 9.10 in YS) We consider the solution of a linear system $Ax = b$ using the pre-conditioned GMRES method with some pre-conditioner M . In the lecture we have discussed left- and right-pre-conditioning for this method, and we have also briefly discussed a general convergence result for the GMRES method.

- a) Show that the matrices AM^{-1} and $M^{-1}A$ have the same eigenvalues. How are the eigenvectors of the two matrices related to each other?

- b) Using the results of part a), would you expect that the left- and the right-pre-conditioned iterations converge:

1. ... in exactly the same number of steps?
2. ... in roughly the same number of steps?

3. ...in roughly the same number of steps provided that the system is not ill-conditioned?

3 We consider once again the solution of the one-dimensional Poisson problem

$$\begin{aligned} -u''(x) &= f(x), \quad 0 < x < 1, \\ u(0) &= u(1) = 0, \end{aligned}$$

discretised using finite differences on a uniform grid with step-size $h = 1/n$. Denote (again) the discretised equations as a system $Au = b$, where A is the discretised second derivative.

- a) Suppose that we apply the CG method in order to solve this problem. Estimate how many steps will be needed in order to reduce the initial error (measured in the A -norm) by 5 orders of magnitude.
- b) We now solve the same problem using the pre-conditioned CG method, and we choose as pre-conditioner two iterates of the Jacobi method. Verify that this pre-conditioner is symmetric, and estimate how many steps of the pre-conditioned method will be necessary to reduce the initial error by 5 orders of magnitude.
- c) Determine whether this type of pre-conditioning makes sense in this situation.

Hint: If two matrices A and B have the same eigenvectors v_i with eigenvalues λ_i and μ_i , respectively, then the matrix AB will have the eigenvalues $\lambda_i\mu_i$. (Also, we have $AB = BA$). Also, if p is a polynomial, then the eigenvalues of $p(A)$ are the values $p(\lambda_i)$.