Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4205 Numerical Linear Algebra Fall 2017

Exercise set 4

1 Assume that $A \in \mathbb{R}^{m \times n}$ with $m > n$ has full rank and that $b \in \mathbb{R}^m$. Consider the following iteration (the *CGNR-method*):

Set $x \leftarrow 0$, $r \leftarrow b$, $z \leftarrow A^T r$, $p \leftarrow z$, $s \leftarrow \|z\|_2^2$;
while *not yet converged* **do**

$$
\begin{aligned}
w &\leftarrow Ap\,;\\
\alpha &\leftarrow s/\|w\|_2^2\,;\\
x &\leftarrow z + \alpha p\,;\\
r &\leftarrow r - \alpha w\,;\\
z &\leftarrow A^T r\,;\\
s_{\text{old}} &\leftarrow s\,;\\
s &\leftarrow \|z\|_2^2\,;\\
\beta &\leftarrow s/s_{\text{old}}\,;\\
p &\leftarrow z + \beta p\,;
\end{aligned}
$$

end

a) Show that this algorithm converges to a solution of the least squares problem

$$\min_x \|Ax - b\|_2.$$

b) Denote by x_k the result after k iterations of this algorithm. Show that x_k minimises $\|Ax - b\|_2$ among all vectors $x \in \mathcal{K}_k(A^T A, A^T b)$.

Hint: Show that this algorithm implements the CG-method for the normal equations.

2 (Cf. Exercise 9.10 in YS) We consider the solution of a linear system $Ax = b$ using the pre-conditioned GMRES method with some pre-conditioner M. In the lecture we have discussed left- and right-pre-conditioning for this method, and we have also briefly discussed a general convergence result for the GMRES method.

a) Show that the matrices AM^{-1} and $M^{-1}A$ have the same eigenvalues. How are the eigenvectors of the two matrices related to each other?

b) Using the results of part **a)**, would you expect that the left- and the right-pre-conditioned iterations converge:

1. . . . in exactly the same number of steps?
2. . . . in roughly the same number of steps?

3. ...in roughly the same number of steps provided that the system is not ill-conditioned?

3 We consider once again the solution of the one-dimensional Poisson problem

$$-u''(x) = f(x), \quad 0 < x < 1,$$
$$u(0) = u(1) = 0,$$

discretised using finite differences on a uniform grid with step-size $h = 1/n$. Denote (again) the discretised equations as a system $Au = b$, where A is the discretised second derivative.

a) Suppose that we apply the CG method in order to solve this problem. Estimate how many steps will be needed in order to reduce the initial error (measured in the A-norm) by 5 orders of magnitude.

b) We now solve the same problem using the pre-conditioned CG method, and we choose as pre-conditioner two iterates of the Jacobi method. Verify that this pre-conditioner is symmetric, and estimate how many steps of the pre-conditioned method will be necessary to reduce the initial error by 5 orders of magnitude.

c) Determine whether this type of pre-conditioning makes sense in this situation.

Hint: If two matrices A and B have the same eigenvectors v_i with eigenvalues λ_i and μ_i, respectively, then the matrix AB will have the eigenvalues $\lambda_i\mu_i$. (Also, we have $AB = BA$). Also, if p is a polynomial, then the eigenvalues of $p(A)$ are the values $p(\lambda_i)$.