Norwegian University of Science
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TMA4205 Numerical Linear Algebra Fall 2017

Exercise set 3

1 (Cf. Exercise 6.8 in Saad.) Consider the solution of the linear system $Ax = b$ with initial guess x_0, where

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad x_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

a) Compute the matrices V_m and H_m, $m = 1, \ldots, 5$, resulting from the application of Arnoldi process.

b) Compute the FOM iterates y_m, x_m, $m = 1, \ldots, m$ (when possible).

c) Describe in detail the QR factorization of the matrices $\tilde{H}_m$, $m = 1, \ldots, 5$, using Givens rotations.

d) Compute the GMRES iterates y_m, x_m, $m = 1, \ldots, m$ (when possible).

2 Assume that $A \in \mathbb{R}^{n \times n}$ is SPD and that we use the CG method for solving the system $Ax = b$. Assume moreover that the eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$ are distributed in an interval $[\lambda_{\min}, \lambda_{\max}] \subset \mathbb{R}_{>0}$, while the eigenvalue λ_n is "very different" from the others (that is, either much larger than $\lambda_{\max}$ or much closer than $\lambda_{\min}$ to 0).

Find an estimate for the error reduction $\|x_m - x^*\|_A / \|x_0 - x^*\|_A$ after m steps of the CG method. Here $x^* = A^{-1}b$ is the exact solution of the system. The estimate should only depend on $\lambda_{\max}$, $\lambda_{\min}$, λ_n, and m.

3 (Cf. Problem 3a, exam 2016.) We are given a linear system of the form

$$(I + uu^T)x = b,$$

where $I \in \mathbb{R}^{n \times n}$ is the n-dimensional identity matrix and $u \in \mathbb{R}^{n \setminus \{0\}}$ is some given non-zero vector. Assume we apply the CG-method for solving this system. How many iterations do you expect the method to take until convergence is reached?