NTNU
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Department of Mathematical Sciences

Examination paper for **TMA4205 Numerical Linear Algebra**

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Permitted examination support material: C: Specified, written and handwritten examination support materials are permitted. A specified, simple calculator is permitted. The permitted examination support materials are:

- Y. Saad: Iterative Methods for Sparse Linear Systems. 2nd ed. SIAM, 2003 (book or printout).
- L. N. Trefethen and D. Bau: Numerical Linear Algebra, SIAM, 1997 (book or photocopy).
- G. Golub and C. Van Loan: Matrix Computations. 3rd ed. The Johns Hopkins University Press, 1996 (book or photocopy).
- E. Rønquist: Note on The Poisson problem in $\mathbb{R}^2$: diagonalization methods (printout).
- M. Grasmair: The singular value decomposition (printout).
- Rottmann, Matematisk formelsamling.
- Your own lecture notes from the course (handwritten).

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Problem 1 Let

$$A = \begin{pmatrix} 6 & 3 \\ 3 & 4 \end{pmatrix}.$$

Perform one step of the QR-method (for computing eigenvalues) with the shift parameter $\mu = 2$.

Problem 2 Assume that $A \in \mathbb{R}^{m \times n}$ with $m < n$ has the non-zero singular values $\sigma_1, \dots, \sigma_m$.

a) Given $\lambda \geq 0$, we define the matrices

$$G_\lambda := \begin{pmatrix} \lambda I & A \\ A^T & -\lambda I \end{pmatrix} \qquad \text{and} \qquad H_\lambda := \begin{pmatrix} \lambda I & A \\ -A^T & \lambda I \end{pmatrix}.$$

Show that the singular values of the matrices G_λ and H_λ are precisely the values $\sqrt{\sigma_k^2 + \lambda^2}$, $k = 1, \dots, m$, and λ. Additionally, show that both matrices are invertible in case $\lambda > 0$, and determine whether the matrices G_λ or H_λ are positive (semi-)definite.

Given $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$, we want to simultaneously solve the linear systems

$$\begin{aligned} u - Av &= a, \\ A^T u + v &= b. \end{aligned} \tag{1}$$

b) Show that the Jacobi method for the solution of this system converges for all initial values $u^{(0)}$ and $v^{(0)}$ in case $\|A\|_2 < 1$.

c) Given $\mu > 0$, we now apply the iteration

$$\begin{aligned} u^{(k+1)} &= \frac{1}{\mu}\Big(a + (\mu - 1)u^{(k)} + Av^{(k)}\Big), \\ v^{(k+1)} &= \frac{1}{\mu}\Big(b + (\mu - 1)v^{(k)} - A^T u^{(k)}\Big). \end{aligned}$$

For which $\mu > 0$ does this iteration converge to a solution of (1) independent of the starting vectors $u^{(0)}$ and $v^{(0)}$?

How should one choose the parameter μ in order to obtain the fastest convergence?

Problem 3

a) We are given a linear system of the form

$$(I + uu^T)x = b,$$

where $I \in \mathbb{R}^{n\times n}$ is the n-dimensional identity matrix, and $u \in \mathbb{R}^n \setminus \{0\}$ is some given non-zero vector. Assume we apply the CG-method for solving this system. How many iterations do you expect the method to take until convergence is reached? Justify your answer!

b) Consider now in particular the system

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}.$$

Use the CG method with initialisation $x^{(0)} = 0$ for solving this system. Iterate until you have reached convergence.

Problem 4 We consider the two linear systems

$$Ax = b \qquad \text{and} \qquad \tilde{A}\tilde{x} = \tilde{b},$$

where

$$\tilde{A} = QAQ^T \qquad \text{and} \qquad \tilde{b} = Qb$$

for some orthogonal matrix Q. Denote by $x^{(k)}$ the k-th iterate for the GMRES method for solving the system $Ax = b$ with initialisation $x^{(0)} = 0$, and by $\tilde{x}^{(k)}$ the k-th iterate for the GMRES method for solving the system $\tilde{A}\tilde{x} = \tilde{b}$, again with initialisation $\tilde{x}^{(0)} = 0$. Show that $\tilde{x}^{(k)} = Qx^{(k)}$.

Problem 5 We are given a linear system $Ax = b$ and some initial guess $x_0 \in \mathbb{R}^n$ of its solution. Denote by x^{SD} the result of one step of the steepest descent method starting with x_0, and by x^{MR} the result of one step of the MR iteration starting with x_0. Assume now that $x^{SD} = x^{MR}$.

Show that in this case the initial residual $b - Ax_0$ is an eigenvector of A. Conclude that $x^{SD} = x^{MR}$ solves the system $Ax = b$.

Problem 6 Consider the matrix

$$A = \begin{pmatrix} 1 & 0.2 & 1 & -0.2 \\ 0.5 & 1.1 & 0.5 & -1.1 \end{pmatrix}.$$

a) Compute the reduced singular value decomposition of the matrix A.

b) Use the singular value decomposition in order to compute the solution of the problem

$$\min_{x \in \mathbb{R}^4} \|x\|_2^2 \qquad \text{s.t. } Ax = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$