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TMA4205 Numerical Linear Algebra Fall 2016

Solutions to exercise set 4

1 We first consider the system $Ax = b$ with $\mathscr{L} = A\mathscr{K}$. Let

$$\mathscr{K} = \operatorname{span}\{v_1, \dots, v_m\},$$
$$V_m = [v_1 \mid \cdots \mid v_m] \in \mathbb{R}^{n\times m}.$$

In the first case, the approximate solution $\tilde{x}$ to $Ax = b$ must satisfy

$$\tilde{x} - x_0 \in \mathscr{K},$$
$$(AV_m)^{\mathrm{T}}(b - A\tilde{x}) = 0.$$

Next, we consider the system $A^{\mathrm{T}}Ax = A^{\mathrm{T}}b$ with $\mathscr{L} = \mathscr{K}$. The approximate solution must now satisfy

$$\tilde{x} - x_0 \in \mathscr{K},$$
$$V_m^{\mathrm{T}}(A^{\mathrm{T}}b - A^{\mathrm{T}}A\tilde{x}) = 0$$
$$\Updownarrow$$
$$(AV_m)^{\mathrm{T}}(b - A\tilde{x}) = 0.$$

Hence, we get the same system $(AV_m)^{\mathrm{T}}A\tilde{x} = (AV_m)^{\mathrm{T}}b$ or $(AV_m)^{\mathrm{T}}A\delta = (AV_m)^{\mathrm{T}}r_0$, where $\tilde{x} = x_0 + \delta$. This shows that the two methods are equivalent.

2 We use $N = 100$, $v = \mathrm{e}_1$, and $m = 10, 20, 30, 40, 50$, as suggested. The results for both **a)** and **b)** are reported in Table 1, where we report the errors $\|V_m^{\mathrm{T}}AV_m - H_m\|_\infty$ and $\|V_m^{\mathrm{T}}V_m - \mathrm{I}_m\|_\infty$ after $m = 10, 20, 30, 40, 50$ iterations for Gram–Schmidt (GS) orthogonalization and modified Gram–Schmidt (MGS) orthogonalization. From the table we conclude that MGS performs slightly better than GS in this case.

	GS		MGS	
m	$\|V_m^{\mathrm{T}}AV_m - H_m\|_\infty$	$\|V_m^{\mathrm{T}}V_m - \mathrm{I}_m\|_\infty$	$\|V_m^{\mathrm{T}}AV_m - H_m\|_\infty$	$\|V_m^{\mathrm{T}}V_m - \mathrm{I}_m\|_\infty$
10	$2.90\cdot10^{-13}$	$3.91\cdot10^{-14}$	$9.69\cdot10^{-14}$	$1.92\cdot10^{-14}$
20	$1.33\cdot10^{-11}$	$1.72\cdot10^{-12}$	$3.89\cdot10^{-12}$	$7.73\cdot10^{-13}$
30	$2.04\cdot10^{-8}$	$2.64\cdot10^{-9}$	$5.98\cdot10^{-9}$	$1.19\cdot10^{-9}$
40	$3.55\cdot10^{-3}$	$4.61\cdot10^{-4}$	$1.04\cdot10^{-3}$	$2.07\cdot10^{-4}$
50	80.4	8.04	19.2	2.41

Table 1: MGS is a little bit better than GS.

3 After applying m steps of Arnoldi process to a matrix A we obtain the matrix V_m containing the orthonormal basis for $\mathcal{K}_m(A, v_1)$ and an upper Hessenberg matrix H_m satisfying the equality $H_m = V_m^T A V_m$. Assuming that $A^T = -A$, the matrix on the right hand side of the equality sign is anti-symmetric. Therefore H_m is also antisymmetric and thus has only two non-zero diagonals:

$$H_m = \begin{pmatrix} 0 & -h_{2,1} & \dots & 0 \\ h_{2,1} & 0 & \ddots & 0 \\ 0 & \ddots & \ddots & -h_{m,m-1} \\ 0 & \dots & h_{m,m-1} & 0 \end{pmatrix} \qquad (1)$$

As a result, Arnoldi process simplifies to (note: only the sub-diagonal of H is computed):

$v_1 := v/\|v\|_2$, $v_0 := 0$, $h_{1,0} := 0$
for j=1,…,m **do**
    $w_j := Av_j + h_{j,j-1} v_{j-1}$
    $h_{j+1,j} := \|w_j\|_2$
    if $h_{j+1,j} = 0$ **then stop**
    end if
    $v_{j+1} := w_j / h_{j+1,j}$
end for