TMA4195 — MATHEMATICAL MODELLING (FALL 2018)
PROJECT DESCRIPTION
GLACIAL MODELLING

1. Introduction

Glaciers form in regions where, on average, more snow falls during winter than melts during summer. As more and more snow accumulates, the pressure within the snow increases and transforms it to glacial ice. If the pressure becomes sufficiently high and the ice is situated on a downward slope, it will eventually begin to flow in the form of a glacier. The speed of the glacier depends on its internal structure, temperature, and depth, and the slope of the underlying bedrock, amongst others, and typically varies between a few metres per year and a few metres per day; the fastest flowing glaciers reach velocities of up to 50 metres per day. Currently, a large number of glaciers are retreating because of the changing climate conditions. An example of this is Engabreen in Nordland, which has decreased significantly in size over the last hundred years, with the most significant changes happening between 1930 and 1950 (see Figure 1 for a comparison).

Figure 1. *Left:* Engabreen in Meløy, Nordland, in 1890. Photo: Library of Congress. *Right:* Engabreen on 29th July 2015. Photo: Hallgeir Elvehøy. Both photos were downloaded from `https://www.nve.no/hydrology/glaciers/glacier-monitoring/engabreen/`.

The goal of this project is to set up a model for the dynamics of a glacier. With this model we will try to describe simultaneously the shape of the surface of the glacier as well as the internal dynamics, as ice slowly flows from the top of the glacier to its toe. As a consequence, we will be able to simulate with relatively simple tools both the development of the shape and size of the glacier and the movement that is happening in its interior.

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2. Ice-sheet modelling

We assume that the glacier flows along a valley of constant, relatively small slope of angle α. By x^* we denote a position along the length of the valley, and by z^* we denote the height above the valley (measured perpendicular to the valley floor). In order to simplify the following analysis, we will neglect all dependence of the y^*-coordinate and assume that the glacier behaves uniformly along its width.

By $h^* = h^*(x^*, t^*)$ we denote the height of the glacier above the valley at a position x^* and a time t^*. A point in the glacier moves at speed $\mathbf{w}^* = (u^*, v^*)$, with u^* denoting the x^* component of the velocity, and v^* the z^* component. The velocity $\mathbf{w}^*$ will depend on time t^* and both the x^* and z^* coordinate.

A rough sketch of our model of the glacier is depicted in Figure 2.

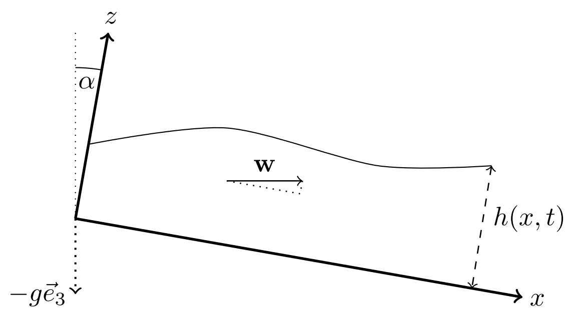

Figure 2. Sketch of the glacier; the angle α denotes the slope of the valley, $h = h(x,t)$ the glacier's height above the valley, and $\mathbf{w} = (u, v)$ its velocity in x and z direction.

In the following, we will formulate equations based on the conservation of mass and momentum that describe the relation between the height of the glacier and the velocity within it. We start by looking at the conservation of mass within the glacier, which can be based on the assumption that the glacial ice has a constant density ρ.

Problem 1. *Explain how the assumption of a constant density ρ within the glacier implies that*

$$\nabla \cdot \mathbf{w}^* = 0.$$

Conservation of momentum and Newton's second law of motion imply that the velocity of the glacier satisfies the equations

$$\begin{aligned} \rho \partial_{t^*} u^* + \rho \nabla u^* \cdot \mathbf{w}^* &= -\partial_{x^*} p^* + \nabla \cdot \tau_x^* + f_x, \\ \rho \partial_{t^*} v^* + \rho \nabla v^* \cdot \mathbf{w}^* &= -\partial_{z^*} p^* + \nabla \cdot \tau_z^* + f_z. \end{aligned} \tag{1}$$

Here $f = (f_x, f_z)$ denote the body forces that are acting on each point, p^* denotes the pressure, and

$$\tau^* = \begin{pmatrix} \tau_x^* \\ \tau_z^* \end{pmatrix} = \begin{pmatrix} \tau_{xx}^* & \tau_{xz}^* \\ \tau_{zx}^* & \tau_{zz}^* \end{pmatrix}$$

the stress tensor. Furthermore, the pressure p^* can be written as a sum

$$p^*(x^*, z^*, t^*) = \rho g \cos\alpha (h^*(x^*, t^*) - z^*) + \tilde{p}^*(x^*, z^*, t^*) \tag{2}$$

with $g \approx 9.8\,\mathrm{m/s^2}$ denoting the gravitational acceleration at the surface of the earth. The first term in (2) describes (approximately) the hydrostatic pressure within the glacier, and the second term the counterpressure that is induced by the assumption of a constant density ρ.

Problem 2. *Model the body forces within the glacier and explain why the left hand side of the equations* (1) *can be set to zero. Show that this results in the simplified equations*

$$\begin{aligned} \nabla \cdot \tau_x^* + \rho g \sin\alpha - \rho g \cos\alpha \partial_{x^*} h^* - \partial_{x^*} \tilde{p}^* &= 0, \\ \nabla \cdot \tau_z^* - \partial_{z^*} \tilde{p}^* &= 0. \end{aligned} \tag{3}$$

The next step will be the development of a reasonable model for the stress tensor τ^*. In ice-sheet modelling, a common assumption is that this stress tensor is related to the strain rate (the deformation rate of the material) by the following relations (*Glen's law*):

$$\begin{aligned} \partial_{x^*} u^* &= \mu(\theta^*)^{m-1} \tau_{xx}^*, \\ \partial_{z^*} v^* &= \mu(\theta^*)^{m-1} \tau_{zz}^*, \\ \frac{1}{2}\big(\partial_{z^*} u^* + \partial_{x^*} v^*\big) &= \mu(\theta^*)^{m-1} \tau_{xz}^* = \mu(\theta^*)^{m-1} \tau_{zx}^*. \end{aligned}$$

Here,

$$\theta^* := \Big(\frac{1}{2}{\tau_{xx}^*}^2 + {\tau_{xz}^*}^2 + \frac{1}{2}{\tau_{zz}^*}^2\Big)^{1/2},$$

and μ and m are material constants depending amongst others on the temperature of the ice. Common values for the exponent m lie between $m = 1.8$ and $m = 5$, with $m = 3$ being a typical choice.

Problem 3. *Simplify the relations between the different entries of the stress tensor by using the incompressibility of the ice.*

After this simplification, we end up with the equations

$$\begin{aligned} \partial_{x^*} u^* + \partial_{z^*} v^* &= 0, \\ \partial_{x^*} \tau_{xx}^* + \partial_{z^*} \tau_{xz}^* + \rho g \sin\alpha - \rho g \cos\alpha \partial_{x^*} h^* - \partial_{x^*} \tilde{p}^* &= 0, \\ \partial_{x^*} \tau_{xz}^* - \partial_{z^*} \tau_{xx}^* - \partial_{z^*} \tilde{p}^* &= 0, \\ \partial_{x^*} u^* &= \mu(\theta^*)^{m-1} \tau_{xx}^*, \\ \frac{1}{2}(\partial_{z^*} u^* + \partial_{x^*} v^*) &= \mu(\theta^*)^{m-1} \tau_{xz}^*, \\ \theta^* &= \big({\tau_{xx}^*}^2 + {\tau_{xz}^*}^2\big)^{1/2}, \end{aligned}$$

for the unknowns u^*, v^*, h^*, $\tilde{p}^*$, τ_{xx}^*, τ_{xz}^*, and θ^* in the interior of glacier. That is, the equations above are valid at the points (x^*, z^*) within the region

$$\Omega(t^*) := \big\{(x^*, z^*) \in \mathbb{R}^2 : 0 < z^* < h^*(x^*, t^*)\big\}.$$

Problem 4. *What can we say about the boundary conditions?*

As a final step in the development of a model for the glacier, we will have to find equations for the surface h^* of the glacier. Again, these equations can be based on the conservation of mass. However, we will have to factor in the accumulation and/or melting of ice that occurs at the surface. Let us assume that this happens at a rate $q(x^*, t^*)$ depending on position x^* and time t^*, but independent of the glacier itself.

Problem 5. *Use the conservation of matter on a "control interval"* $[x^*, x^* + \varepsilon]$ *in* x^**-direction to derive the equation*

$$\frac{\partial h^*}{\partial t^*} + \frac{d}{dx^*} \int_0^{h^*(x^*, t^*)} u^*(x^*, z^*, t^*)\, dz^* = q(x^*, t^*) \tag{4}$$

coupling the height of the glacier with its velocity in x^* *direction.*

3. Shallow ice approximation

In the previous section, we have derived a general set of equations that describe the behaviour of glaciers. However, the specific geometry of a glacier has not yet been taken into account. In the following, we will do so by rescaling the equations in an appropriate way and identifying (and later ignoring) small terms. The main basis for the scaling is the fact that the length L of a typical glacier is significantly larger than its height H. We can therefore choose the scales

$$x^* = Lx \qquad \text{and} \qquad z^* = Hz,$$

for the x and z coordinates, and try to make use of the fact that

$$\varepsilon := \frac{H}{L} \ll 1.$$

Moreover, it makes sense to scale the height h^* of the glacier with the same scale H and set

$$h^* = Hh.$$

Next, we will try to find good scales U and V for the velocities in the glacier, and a scale T for the time.

Problem 6. *Assume that Q is a typical value for the accumulation rate $q(x^*, t^*)$ of the ice. Suggest scales $u^* = Uu$, $v^* = Vv$, and $t^* = Tt$ such that the terms in the equations for the conservation of matter balance.*

Finally, we will need scales Θ, Θ_{xx}, and Θ_{xz} for the stresses θ^*, τ^*_{xx}, and τ^*_{xz}, and a scale P for the pressure $\tilde{p}^*$. Here we will use the scales

$$\Theta = \Theta_{xz} = \rho g H \sin\alpha \qquad \text{and} \qquad \Theta_{xx} = P = \frac{H}{L}\Theta,$$

which will balance the other equations in our system.

We can now rescale the equations and ignore terms of order $\varepsilon = H/L$. This will allow us some major simplifications and, eventually, a formulation for the surface behaviour only. As a first step, it is possible to obtain relatively simple expressions for the z-derivatives of the shear stress τ_{xz} and the velocity u.

Problem 7. *Rescale the equations for the glacier and perform an asymptotic expansion with respect to the parameter ε. Show in particular that this results in the relations*

$$\partial_z \tau_{xz} \approx -1 \qquad \text{and} \qquad \partial_z u \approx \kappa |\tau_{xz}|^{m-1}\tau_{xz}$$

for some constant κ.

Because of this very simple form of the z-derivatives of the stress τ_{xz} and the velocity u, it is possible to derive explicit formulas for these functions depending only on the height $h(x,t)$ of the glacier and the values of τ_{xz} and u at the surface and bottom of the glacier, respectively. The shear stress τ_{xz} at the surface $h(x,t)$ is obviously equal to zero. Moreover, we will assume in the following that the velocity at the bottom of the glacier is also equal to zero.[1]

Problem 8. *Use the boundary conditions $\tau_{xz}(h) = 0$ and $u(x,0,t) = 0$, and the (rescaled) equation* (4) *in order to derive the equation*

$$\partial_t h + \lambda \frac{d}{dx} h^{m+2} = q, \tag{5}$$

[1] This assumes that we are dealing with "cold glaciers", where the base is frozen to the ground. It is also possible, that the base of the glacier has a temperature that is actually above the melting point of ice, especially as that melting point decreases with increasing pressure. Then it is possible that the glacier actually flows on a (relatively) thin water film, in which case the base velocity $u(x,0,t)$ might be larger than zero.

and determine the constant λ. What are the boundary conditions for this equation?

Problem 9. *Building upon the method used in Problem 8, derive explicit formulas (in dependence of h) for the velocities u and v within the glacier.*

Hint (Problems 8 and 9)**.** Use the approximations derived in Problem 7 in order to obtain explicit expressions for τ_{xz} and u (depending on h). Insert these in (4) (after rescaling), and finally use the conservation of mass within the glacier in order to obtain an expression for v.

All the computations above have assumed that the bottom of the valley where the glacier flows is flat, and its inclination α is constant. If these assumptions are not satisfied, it is necessary to adjust the previous equations.

Problem 10. *Assume that the valley has a non-flat height profile $d(x)$ with $0 < d(x) < h(x,t)$. How can one incorporate this into the equations derived above?*

An implicit assumption in the derivation of the simplified equations is that the slope of the valley is relatively steep and that its slope α should not be treated as a small parameter in the same way as ε. However, it is also possible to treat both ε and α simultaneously as small parameters and still obtain a relatively simple PDE for h.

Problem 11. *Assume that the parameter $\gamma := \varepsilon \cot \alpha$ is approximately of order 1. How does this change the asymptotic expansion derived in Problem 7, and what are the consequences for the equation* (5)*?*

4. Dynamics of glaciers

Above, we have derived equations that describe approximately the behaviour of a glacier. In the following, we will use these equations for answering some specific questions. First we will concentrate on "stationary glaciers", where the flow of the glacier perfectly balances the accumulation and melting of ice in such a way that the shape of the glacier does not change over time.

Problem 12. *Assume that the accumulation rate q is independent of time. Compute possible steady states of the glacier. What are reasonable boundary conditions? At which point lies the toe x_F of the glacier?*

Problem 13. *What happens, if the steady state of a glacier is slightly perturbed? Which parts of the glacier react most strongly to such perturbations?*

Hint. Start with a steady state $h_0 = h_0(x)$ for some given accumulation rate $q = q(x)$ and assume a small perturbation $h_\delta = h_0 + \delta k$, where $\delta > 0$ is a small parameter. Insert the perturbed function in the PDE (5) and show that this results in an equation of the form

$$\delta \partial_t k + \delta \frac{d}{dx} g(h,k) + o(\delta) = 0.$$

Ignore the terms of order $o(\delta)$ and try to analyse (either qualitatively or numerically for specific situations) the resulting hyperbolic equation for k.

Although the glacier appears to be stationary, all parts of it are in constant movement: Snow[2] that falls on the top part of the glacier will over time vanish and be slowly transported through the interior of the glacier, until it appears again (and melts) near the toe of the glacier.

[2] And other objects like airplanes — see e.g. `https://en.wikipedia.org/wiki/1946_C-53_Skytrooper_crash_on_the_Gauli_Glacier`

Problem 14. *Implement a method that computes numerically the flow field within a stationary glacier for a given accumulation rate q.*

Since it is almost impossible to obtain accurate measurements of the accumulation rate of snow in a glacier, we will use a very simple model for further computations, which is based on the following assumptions:

- Above the permanent snow line, at a point x_S in the valley, we have a constant positive accumulation rate q_0. Here, almost all precipitation takes the form of snow, and we have only moderate snow melting during summer.
- Starting at the point x_S, we observe rainfall during summer and the rate at which snow melts increases, resulting in a decrease of q. For simplicity, we assume that this decrease is linear.
- Finally, after the toe of the glacier at a point x_F, the accumulation rate is equal to zero.

The resulting model of the accumulation rate q is sketched in Figure 3.

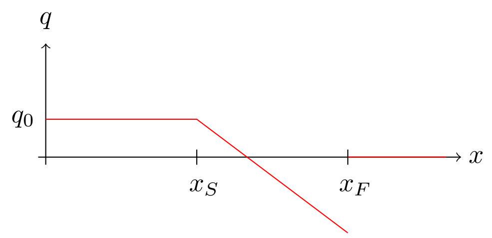

FIGURE 3. Sketch of the situation described above: Above the permanent snow line x_S, the accumulation rate is constant; then it decreases steadily as the average temperatures rise; after the toe of the glacier at x_F, the accumulation rate is equal to zero.

Problem 15. *Calculate in particular the flow field and possible trajectories through the glacier for accumulation rates q as depicted in Figure 3, and present your results.*

Finally, we can implement a numerical method for the equation (5) (or the modification obtained in Problem 11) and thus simulate the behaviour of the glacier under changing climate conditions.

Problem 16. *Implement a numerical method for the solution of the hyperbolic equation* (5) *and/or the parabolic equation derived in Problem 11. Use your method in particular to simulate the change of a glacier that occurs if the permanent snow line rises due to increasing average temperatures.*