NTNU
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Examination paper for **TMA4195 Mathematical Modelling**

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Permitted examination support material:

- Rottmann, Mathematical formulae.
- Approved basic calculator.

Other information:

- You may answer to the questions of the exam either in English or in Norwegian.
- Good luck!

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Informasjon om trykking av eksamensoppgave	
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skal ha flervalgskjema ☐	

Date Signature

Problem 1 Consider the initial value problem

$$\begin{aligned} m\ddot{y}^* &= -\lambda(\dot{y}^*)^3 - ky^*, \qquad (1)\\ y^*(0) &= 0,\\ \dot{y}^*(0) &= v_0, \end{aligned}$$

which models a spring pendulum of mass m, spring constant k, and cubic damping with damping parameter λ, which is perturbed from its equilibrium position with initial velocity $v_0 \neq 0$.

Propose a rescaling for this equation that is valid for times close to zero based on the assumption that the first two terms in (1) (acceleration and damping) dominate. Under which condition does this yield a reasonable scaling?

Possible solution

- After rescaling with scales $y = Yy^*$ and $t = Tt^*$, we will obtain the ODE

$$\begin{aligned} \frac{mY}{T^2}\ddot{y} &= -\frac{\lambda Y^3}{T^3}\dot{y}^3 - kYy,\\ y(0) &= 0,\\ \frac{Y}{T}\dot{y}(0) &= v_0. \end{aligned}$$

At times close to zero, we can expect the velocity to be close to v_0 and the position to be approximately 0. In order to obtain a well-scaled velocity, we should choose Y and T such that

$$\frac{Y}{T} = v_0.$$

Balancing the first two terms then yields the condition

$$\frac{mY}{T^2} = mv_0\frac{1}{T} \approx \lambda\frac{Y^3}{T^3} = \lambda v_0^3.$$

This suggests the time scale

$$T = \frac{mv_0}{\lambda v_0^3} = \frac{m}{\lambda v_0^2}.$$

For the spatial scale Y this implies

$$Y = Tv_0 = \frac{m}{\lambda v_0}.$$

As a consequence, we obtain the rescaled equation

$$\ddot{y} = -\dot{y}^3 - \kappa y \qquad \text{with} \qquad \kappa = \frac{km}{\lambda v_0^4}.$$

As long as the parameter κ is small or at most of order 1, this scaling makes sense. If, however, $\kappa \gg 1$ (we have a high initial velocity and low damping), the basic assumption that $\ddot{y} \sim \dot{y}$ can only hold if $y \ll 1$. Thus either the balancing assumption is incorrect in that case, or the solution is badly scaled.

Problem 2 Consider the differential equation

$$\epsilon y'' + y' = \frac{y + y^3}{1 + 3y^2}$$

with boundary conditions $y(0) = 0$ and $y(1) = 1$. Find leading order outer, inner and uniform solutions for small $\epsilon > 0$ using the fact that there is a boundary layer at $x = 0$.

Hint: It is enough to provide an implicit form for the outer solution.

Possible solution

- For the outer solution, we have to solve the equation

$$y' = \frac{y + y^3}{1 + 3y^2}.$$

This can be rewritten as

$$\frac{1 + 3y^2}{y + y^3} dy = dx,$$

which integrates to

$$\ln(y(1 + y^2)) = C + x$$

for some constant C, which is determined by the boundary condition $y(1) = 1$. A simple computation yields

$$C = \ln 2 - 1.$$

That is, the outer solution is the unique solution of the equation

$$y_o(x)(1 + y_o(x)^2) = 2e^{x-1}.$$

- Next we need to find a correct scale for the boundary layer. To that end we rescale the equation by setting $x = \delta\xi$ and set $Y(\xi) := y(x)$. After rescaling, we will obtain
$$\frac{\epsilon}{\delta^2}Y'' + \frac{1}{\delta}Y' = \frac{Y + Y^3}{1 + 3Y^2}.$$
We try to match the terms on the right hand side setting $\delta = \epsilon$ and obtain the rescaled equation
$$Y'' + Y' = \epsilon\frac{Y + Y^3}{1 + 3Y^2}.$$

- Next we solve the approximation to the inner equation
$$Y'' + Y' = 0$$
with boundary condition $Y(0) = 0$. The general solution is of the form
$$Y(\xi) = D + Ee^{-\xi}.$$
With the boundary condition $Y(0) = 0$, we obtain
$$Y(\xi) = D(1 - e^{-\xi})$$

- In order to match the solutions, we have to choose D such that
$$\lim_{\xi\to\infty} Y(\xi) = \lim_{x\to 0^+} y(x).$$
We have
$$\lim_{\xi\to\infty} Y(\xi) = D$$
and
$$\lim_{x\to 0^+} (\ln(y(x)(1 + y(x)^2))) = C = \ln 2.$$
That is,
$$\lim_{x\to 0^+} y(x)(1 + y(x)^2) = 2$$
and thus
$$\lim_{x\to 0^+} y(x) = 1.$$
As a consequence,
$$D = 1,$$
and we have the inner solution
$$Y(\xi) = 1 - e^{-\xi}$$
and the uniform solution
$$y(x) = y_o(x) - e^{-x/\epsilon}$$
where $y_o(x)$ solves
$$y_o(x)(1 + y_o(x)^2) = 2e^{x-1}.$$

Problem 3 The maximal power P that a wind turbine can produce depends on the density ρ of the air, the air velocity v, and the length r of its rotor blades. The dimensions of these physical quantities are $[P] = \mathrm{kg\,m^2/s^3}$, $[\rho] = \mathrm{kg/m^3}$, $[v] = \mathrm{m/s}$, and $[r] = \mathrm{m}$. Find the most general dimensionally consistent model for the power P depending on the other physical quantities mentioned above.

Wind turbines start to operate only when a minimum output power is reached. For air densities of approximately $\rho_E \approx 1.2\mathrm{kg/m^3}$, which are typical at the surface of the earth at medium temperatures, this happens at wind speeds of about $v_E \approx 4\mathrm{m}/s$.
In contrast, the air density on the surface of Mars is about $\rho_M \approx 2 \cdot 10^{-2}\mathrm{kg/m^3}$, and wind speeds are about 8m/s during a typical day, and about 20m/s during a storm. Would a standard terrestian wind turbine operate during a typical martian day or during a martian storm?

Possible solution

- We can assume a relation $P = P(\rho, v, r)$ between the power P and the other quantities. Since the only term on the right hand side that involves kg is the density ρ, the dimensional consistency of the relation implies that $P = \rho F(v, r)$ for some function F. Next, we observe that $[F(v, r)] = [P/\rho] = \mathrm{m^5/s^3}$. The only dimensionally consistent combination of v and r with this dimension is $F(v, r) = Kv^3r^2$ for some constant K. Thus

  $$P = K\rho v^3 r^2.$$

  The atmosphere on Mars is about 60 times less dense than that of the earth. In order to obtain the same power output, the wind speed has therefore to be about $\sqrt[3]{60} \approx 4$ times larger than on earth. That is, if a wind turbine requires a minimum wind speed of 4m/s to work on earth, it would require a wind speed of 16m/s on Mars. In other words, during a typical day it would not work, but it would be perfectly fine during a storm.

Remarks

- The wind speeds given for Mars in this exercise are not completely accurate in that they show large seasonal variations. During the Phoenix mission, average wind speeds were at first consistently around 4m/s, while they increased to the cited 8m/s only in the later stages of that mission (see [HRGM$^+$10]).

Thus the chances for a green revolution on the red planet based on clean wind energy are even slimmer than what the results of this problem indicate.

Problem 4 In order to control insect numbers, it has been suggested to maintain a stable number of sterile male insects in the population. The main idea is that the sterile males compete with fertile males over females; however, if females mate with the sterile males, no offspring is produced. This can effectively reduce the reproduction rate of the insects.

We consider now specifically a model where the population of insects is described, after rescaling, by the equation

$$\frac{\partial N}{\partial t} = \frac{N^2}{N+S} - (\kappa + N + S)N$$

for some parameter $0 < \kappa < 1$.

Determine the equilibrium states of the insect population as a function of S, and discuss the stability of these equilibria. What is the smallest non-zero stable population of insects that can be achieved according to this model?

Possible solution

- Write $f(N,S) = \Big(\frac{N}{N+S} - \kappa - N - S\Big)N$ the right hand side of the ODE. Then the equilibrium points of this ODE are the states $N = 0$ and the solutions (if existent) of the equation $N/(N+S) - \kappa - N - S = 0$. The latter can be rewritten as $N^2 + (\kappa + 2S - 1)N + \kappa S + S^2 = 0$, which has the solutions

  $$N = \frac{1-\kappa-2S}{2} \pm \frac{1}{4}\sqrt{(1-\kappa-2S)^2 - 4\kappa S - 4S^2}.$$

  The term in the root can be simplified to $(1-\kappa)^2 - 4S$. Thus we have

  $$N_\pm = \frac{1-\kappa-2S}{2} \pm \frac{1}{2}\sqrt{(1-\kappa)^2 - 4S}.$$

  As a consequence, we have non-zero equilibrium points if and only if $(1-\kappa)^2 - 4S \geq 0$ or $4S \leq (1-\kappa)^2$. Since $0 < \kappa < 1$, we have always in this case that $1 - \kappa - 2S > 0$ and thus the non-zero equilibrium points are always positive for $S > 0$. For $S = 0$, we have $N_+ = 1 - \kappa$ and $N_- = 0$.

  Thus we have:

- Two positive equilibria at $N_\pm$ for $0 < 4S \leq (1-\kappa)^2$.
- One positive equilibrium at $N_+ = N_-$ for $4S = (1-\kappa)^2$.
- No positive equilibria for $4S > (1-\kappa)^2$.
- Moreover, at the parameter $4S = (1-\kappa)^2$ we have a regular turning point.
- At $S = 0$, we have only one positive equilibrium point $N_+ = 1-\kappa$.

In order to analyse the stability of the equilibria, we compute the derivative of $f(N,S)$ with respect to N:

$$\partial_N f(N,S) = \Big(\frac{N}{N+S} - \kappa - N - S\Big) + \Big(\frac{1}{N+S} - \frac{N}{(N+S)^2} - 1\Big)N.$$

For $N = 0$ and $S > 0$ we obtain $\partial_N f(0,S) = -\kappa - S < 0$. Thus this equilibrium point is asymptotically stable for $S > 0$. Since we have a regular turning point at $4S = (1-\kappa)^2$ (and one can verify that $\partial_N f(N_\pm, S) \neq 0$) it follows that the lower branch of the equilibria (the point N_-) is unstable, whereas the upper branch (the point N_+) is asymptotically stable.

The function $S \mapsto N_+$ is concave, and therefore admits its minimum at the boundary of the admissible interval $[0, (1-\kappa)^2/4]$. For $S = 0$ we obtain a stable population of $N_+ = 1-\kappa$, and for $S = (1-\kappa)^2/4$ we obtain a population of

$$N_+ = \frac{1-\kappa}{2} - \frac{(1-\kappa)^2}{4},$$

which is obviously the smaller of the two possibilities and thus the smallest achievable non-zero population.

Remarks

- The method discussed in this exercise is called *sterile insect technique* (see [DHR05] for a rather large overview). Because of the seasonal dependence of insect populations, they are usually modelled with difference equations instead of differential equations (see [Bar05]); still, the general results of the basic models are qualitatively similar. The basis of the model discussed in this problem is the logistic equation

$$\frac{\partial N}{\partial t} = \alpha N - (\beta + \gamma N)N,$$

which assumes a constant reproduction rate of the insects, but a death rate that increases linearly with the population. The introduction of the sterile insect increases the population pressure and effectively decreases the reproduction rate from α to $\alpha \frac{N}{N+S}$, since only a fraction of the females actually mate with fertile males. This results in the model

$$\frac{\partial N}{\partial t} = \frac{\alpha N}{N+S} N - (\beta + \gamma(N+S))N.$$

A more complicated model that takes into account different stages of the development of the insects and also distinguishes between male and female populations can be found in [ADL12].

Problem 5 This exercise is concerned with the formulation of a traffic flow model for ant trails: As ants move along an established trail, they deposit a *trail pheromone*, which slowly evaporates over time. This trail pheromone serves as an orientation marker for the ants traveling along the trail.

Derive a PDE model for the ant density $\rho(x,t)$ and pheromone concentration $p(x,t)$ along an infinitely long ant trail that is based on the "conservation of ants along the trail" and the following assumptions:

- As each ant moves, it deposits a trail pheromone at a constant rate.
- The trail pheromone evaporates at a constant rate.
- There is a chance that an ant will lose the trail. The rate at which this happens is a function $f(p)$ depending on the pheromone concentration p.
- The speed of the ants is given by

  $$v = v_0(1 + \alpha p)(\rho_{\max} - \rho)$$

  for ant densities smaller than the maximal density $\rho_{\max}$.

Possible solution

- The amount of pheromones that are deposited is proportional to the ant density. Because the evaporation rate is constant as well, the pheromones can be modelled by the equation

  $$\partial_t p(x,t) = \alpha \rho(x,t) - \beta p(x,t)$$

for some constants α, $\beta > 0$. The modelling of ρ is based on the conservation law for formicidae, which states that the change of the total amount of ants in a piece $[a, b]$ of the ant trail is equal to the number of incoming ants through a minus the number of exiting ants through b minus the number of ants lost

The ant flux J is equal to the ants' velocity times their density, that is,

$$J(x,t) = v_0(1 + \alpha p(x,t))(\rho_{\max} - \rho(x,t))\rho(x,t).$$

The loss of ants in the interval $[a, b]$ is given by

$$Q(x, [a,b]) = -\int_a^b f(p(x,t))\rho(x,t)\,dx.$$

Thus we obtain at the integral formulation

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = J(a,t) - J(b,t) - \int_a^b f(p(x,t))\rho(x,t)\,dx.$$

Dividing by $b - a$ and considering the limit $b \to a$ leads to the differential form

$$\partial_t\rho + \partial_x\Big(v_0(1 + \alpha p)(\rho_{\max} - \rho)\rho\Big) = -f(p)\rho.$$

Remarks

- The model discussed in this problem assumes that an ant trail has already formed. However, it is also possible, though much more involved, to model the formation of the trail itself. A PDE based model of the trail formation has for instance been proposed and numerically analysed in [Amo15]. It is also worth mentioning that the modeling of the ant velocity based on standard traffic models might be unrealistic. Experimental studies on *Leptogenys processionalis* indicate that the density of ants on a trail has almost no effect on their velocity [JSCN09].

Problem 6 A satellite that orbits the earth satisfies the system of equations

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -\frac{GM}{r^2}, \\ \frac{d}{dt}\Big(r^2\dot{\theta}\Big) &= 0, \end{aligned}$$

where (r, θ) is the position of the satellite in polar coordinates (centered at the center of the earth), G is the gravitational constant, and M the mass of the earth. One possible solution of this system of equations is given by

$$r = a, \qquad \dot{\theta} = \omega, \qquad \text{with} \qquad a^3\omega^2 = GM.$$

This describes a circular orbit at constant radius a and constant angular velocity ω.

We now consider a small perturbation of that circular orbit and use a regular perturbation of the form $r = a + \epsilon r_1 + \ldots$ and $\dot{\theta} = \omega + \epsilon\dot{\theta}_1 + \ldots$ in order to find a linearisation of this equation around the circular orbit. Find the equations for r_1 and $\dot{\theta}_1$, and verify that the radial term of the linearised solution has the general form

$$r_1(t) = A\sin(\omega t) + B\cos(\omega t) + C$$

and thus remains bounded for all time.

Possible solution

- We write $r = a + \epsilon r_1 + \epsilon^2 r_2 + \ldots$ and $\dot{\theta} = \omega + \epsilon\dot{\theta}_1 + \epsilon^2\dot{\theta}_2 + \ldots$ and insert this series into the equations.

  Ignoring all terms of order ϵ^2 or higher, we obtain for the first equation

  $$(a^2 + 2a\epsilon r_1)\epsilon\ddot{r}_1 - (a^3 + 3a^2\epsilon r_1)(\omega^2 + 2\omega\epsilon\dot{\theta}_1) = -GM + O(\epsilon^2)$$

  or

  $$-a^3\omega^2 + \epsilon\Big(a^2\ddot{r}_1 - 3a^2\omega^2 r_1 - 2\omega a^3\dot{\theta}_1\Big) = -GM + O(\epsilon^2).$$

  Since $a^3\omega^2 = GM$, this simplifies to

  $$\ddot{r}_1 - 3\omega^2 r_1 - 2\omega a\dot{\theta}_1 = 0.$$

  The second equation yields

  $$\frac{d}{dt}\Big((a^2 + 2\epsilon r_1)(\omega + \epsilon\dot{\theta}_1\Big) = O(\epsilon^2),$$

  which simplifies to

  $$\epsilon(2a\dot{r}_1\omega + a^2\ddot{\theta}_1) = O(\epsilon^2)$$

  or

  $$2\dot{r}_1\omega + a\ddot{\theta}_1 = 0.$$

This can be written as

$$a\dot{\theta}_1 = C - 2\omega r_1$$

for some constant C. Inserting this into the first equation yields

$$\ddot{r}_1 - 3\omega^2 r_1 + 4\omega^2 r_1 - 2\omega C = 0$$

or

$$\ddot{r}_1 + \omega^2 r_1 = 2\omega C.$$

The general solution of this equation is

$$r_1 = A\sin(\omega t) + B\cos(\omega t) + 2C/\omega,$$

which remains bounded for all time.

Problem 7 We consider the (scaled) traffic model

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\Big(\rho(1-\rho)\Big) = 0$$

for modeling the traffic on a long single lane road. At position $x = 1$ there is a red traffic light behind which a queue starts to form. At time $t = 0$, when the traffic light turns green, the density of cars is given by

$$\rho_0(x) = \begin{cases} x & \text{if } 0 < x < 1, \\ 0 & \text{else.} \end{cases}$$

Sketch the characteristics of this equation, and show that a shock forms at $(x,t) = (1/2, 1/2)$. In addition, compute the solution of the equation for time $t < 1/2$ and show that the position $s(t)$ of the shock satisfies the differential equation

$$\dot{s} = \frac{s+t-1}{2t}, \qquad s(1/2) = 1/2.$$

Possible solution

- The characteristics satisfy the equation

$$\begin{aligned} \dot{x} &= 1 - 2z, & x(0) &= x_0, \\ \dot{z} &= 0, & z(0) &= \rho_0(x_0), \end{aligned}$$

implying that

$$x(t) = x_0 + (1 - 2\rho_0(x_0))t$$

whenever possible (that is, no characteristics have collided). Specifically, we obtain for the characteristics that start at $x_0 \in [0, 1]$ the formula

$$x(t) = x_0 + (1 - 2x_0)t.$$

Thus two such characteristics starting at points x_0 and x_1 collide when

$$x_0 + (1 - 2x_0)t = x_1 + (1 - 2x_1)t,$$

that is, for $t = 1/2$. Moreover, they all meet at the point $x = 1/2$. As a consequence, we expect a shock to form at $(x, t) = (1/2, 1/2)$. Immediately to the left of the shock, the solution is equal to zero (because of the incoming characteristics from starting points $x_0 < 0$). Immediately to the right of the shock, there is still some work to do...

At the discontinuity at $x = 1$, the characteristics have speed -1 immediately to the left, and speed $+1$ immediately to the right. Thus, a rarefaction wave is forming in the region $|x - 1| > t$ (until we hit the shock). We model the rarefaction wave by

$$\rho(x, t) = \varphi\Big(\frac{x-1}{t}\Big),$$

where we choose φ in such a way that the PDE is satisfied, that is,

$$-\frac{x-1}{t^2}\varphi'\Big(\frac{x-1}{t}\Big) + \frac{1}{t}\Big(1 - 2\varphi\Big(\frac{x-1}{t}\Big)\Big)\varphi'\Big(\frac{x-1}{t}\Big) = 0,$$

or (since $\phi' \neq 0$)

$$1 - 2\varphi(s) = s,$$

that is, $\varphi(s) = (1 - s)/2$, and

$$\rho(x, t) = \frac{t + 1 - x}{2t}.$$

This gives the solution immediately right to the shock.

We now consider the shock itself: The shock $(s(t), t)$ satisfies the Rankine–Hugoniot condition

$$\dot{s} = \frac{[j]}{[\rho]},$$

where $[j] = j^+ - j^-$ denotes the jump in the flux at the shock, and $[\rho] = \rho^+ - \rho^-$ denotes the jump in density along the shock. We have $\rho^- = 0$ and $j^- = 0$, and

$$\rho^+ = \frac{t + 1 - s(t)}{2t}, \qquad j^+ = \rho^+(1 - \rho^+).$$

Thus

$$\frac{[j]}{[\rho]} = \frac{j^+ - j^-}{\rho^+ - \rho^-} = \frac{\rho^+(1-\rho^+)}{\rho^+} = 1 - \rho^+ = \frac{t-1+s}{2t}.$$

Thus the shock satisfies the equation

$$\dot{s} = \frac{t-1+s}{2t}$$

with initial condition $s(1/2) = 1/2$.

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