Background and summary Buckingham's Π theorem

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Definitions from last time

Physical quantity:

$$R = v(R) \cdot [R]$$

All physical quantities have a **dimension** which is intrinsic to them (it describes their nature) examples are

dimension	Mass	Length	Time	Temperature	...	combinations
symbol	M	L	T	θ	...	$\frac{M \cdot L}{T^2}$
units (SI)	Kg	m	s	K	...	$\frac{Kg \cdot m}{s^2}$

Dimensional analysis is the analysis of physical relations (i.e. relations involving physical quantities).

Axioms of dimensional analysis

1. All relations between physical quantities must be dimensionally correct (or consistent).
2. No physical relation should depend on a particular choice of units.

Theorem (Buckingham-Π)
Any *physically meaningful* relation

$$\Phi(R_1, \ldots, R_n) = 0$$

is equivalent to a relation

$$\Psi(\pi_1, \ldots, \pi_{n-r}) = 0$$

involving a maximal set of independent dimensionless quantities. Here $r = \mathrm{rank}(A)$ is the rank of the dimension matrix A.

Dimension matrix

Given $R_1, \ldots, R_n$ physical quantities and the system of units $F_1, \ldots, F_\ell$ such that

$$[R_j] = F_1^{a_{1j}} \cdot \cdots \cdot F_\ell^{a_{\ell,j}}, \quad j = 1, \ldots, n$$

$$A = \begin{array}{c|cccc} & R_1 & \ldots & \ldots & R_n \\ \hline & & & & \\ F_1 & a_{11} & \ldots & \ldots & a_{1,n} \\ \ldots & \vdots & \ldots & \ldots & \vdots \\ F_\ell & a_{\ell,1} & \ldots & \ldots & a_{\ell,n} \end{array}$$

Changes of units

Let $\hat{F}_1, \ldots, \hat{F}_\ell$ be an new set of units such that

$$F_i = x_i \hat{F}_i, \quad x_i > 0, \quad [x_i] = 1$$

then

$$\begin{aligned} R &= v(R)\, F_1^{\alpha_1} \cdot \cdots \cdot F_\ell^{\alpha_\ell} = v(R)\, x_1^{\alpha_1} \cdots x_\ell^{\alpha_\ell}\, \hat{F}_1^{\alpha_1} \cdot \cdots \cdot \hat{F}_\ell^{\alpha_\ell} \\ &= \hat{v}(R)\, \hat{F}_1^{\alpha_1} \cdot \cdots \cdot \hat{F}_\ell^{\alpha_\ell} \end{aligned}$$

and

$$\hat{v}(R) = v(R)\, x_1^{\alpha_1} \cdots x_\ell^{\alpha_\ell}$$

$\Phi(R_1,\dots,R_n)=0$ is physically meaningful iff

1. $\Phi(R_1,\dots,R_n)$ has appropriate units and an appropriate value:

$$
\begin{aligned}
[\Phi(R_1,\dots,R_n)] &= F_1^{\gamma_1}\cdot\cdots\cdots F_\ell^{\gamma_\ell}\\
v(\Phi(R_1,\dots,R_n)) :&= \Phi(v(R_1),\dots,v(R_n))
\end{aligned}
$$

   and so

$$
\begin{aligned}
\Phi(R_1,\dots,R_n)=0 &\Leftrightarrow v(\Phi(R_1,\dots,R_n))[\Phi(R_1,\dots,R_n)]=0\\
&\Leftrightarrow \Phi(v(R_1),\dots,v(R_n))F_1^{\gamma_1}\cdot\cdots\cdot F_\ell^{\gamma_\ell}=0
\end{aligned}
$$

2. Under any change of units $F_i=x_i\hat{F}_i$, $x_i>0$, $[x_i]=1$

$$
\begin{aligned}
\Phi(\hat v(R_1),\dots,\hat v(R_n)) &= \hat v(\Phi(R_1,\dots,R_n))\\
&= v(\Phi(R_1,\dots,R_n))x_1^{\gamma_1}\cdots x_\ell^{\gamma_\ell}\\
&= \Phi(v(R_1),\dots,v(R_n))x_1^{\gamma_1}\cdots x_\ell^{\gamma_\ell}
\end{aligned}
$$

Definitions

$R_1, \ldots, R_n$ are **independent physical quantities** iff the corresponding columns of the dimension matrix are linearly independent vectors.

A **dimensionless combination** of $R_1, \ldots, R_n$ is a combination

$$\pi = R_1^{\lambda_1} \cdots R_n^{\lambda_n},$$

with $\overrightarrow{\lambda} = (\lambda_1, \ldots, \lambda_n) \neq \overrightarrow{0}$, is such that

$$[\pi] = 1.$$

A set of k **dimensionless combinations** of $R_1, \ldots, R_n$

$$\pi_j = R_1^{\lambda_{1j}} \cdots R_n^{\lambda_{nj}}, \qquad j = 1, \ldots, k$$

are **independent** iff the vectors

$$\overrightarrow{\lambda}_j := (\lambda_{1j}, \ldots, \lambda_{nj}), \quad j = 1, \ldots, k$$

are linearly independent.

Theorem Let

A1) $F_1, \ldots, F_\ell$ be a set of fundamental units,

A1) $R_1, \ldots, R_n$ be physical quantities,

A1) $\Phi(R_1, \ldots, R_n) = 0$ be physically meaningful with $R_j \neq 0$ $j = 1, \ldots, n$.

Then

$$\Phi(R_1, \ldots, R_n) = 0$$

is equivalent to a relation

$$\Psi(\pi_1, \ldots, \pi_{n-r}) = 0$$

involving a maximal set of independent dimensionless combinations. Here $r = \mathrm{rank}(A)$ is the rank of the dimension matrix A.

We divide the proof of the theorem in three lemmas, and break it down in simpler tasks.

Assume $R_1, \ldots, R_n$ are given physical quantities and $r = \mathrm{rank}(A)$ then there are precisely $n - r$ independent dimensionless combinations $\pi_1, \ldots, \pi_{n-r}$.
Suggestions for the proof:

$$1 = [\pi] = [R_1^{\lambda_1} \cdots R_m^{\lambda_m}]$$

write this further using the fundamental units and $[R_j] = F_1^{a_{1j}} \cdots F_\ell^{a_{\ell,j}}$ and take logarithms on both sides of the resulting equation. You will obtain a homogeneous system

$$A\overrightarrow{\lambda} = 0.$$

The solutions of this system are the elements of the nullspace of A, $N(A)$. Use a well known result of linear algebra i.e. $\dim(N(A)) = n - r$ (review this result if you have forgotten it!). So there are $n - r$ linearly independent vectors satisfying the homogeneous system, these vectors correspond to the $n - r$ independent dimensionless combinations.

Buckingham-Π: LEMMA 2

Consider the change of units

$$F_i = x_i \hat{F}_i, \qquad x_i > 0, \qquad [x_i] = 1.$$

a) If π is a dimensionless combination of $R_1, \dots, R_n$, then

$$v(\pi) = \hat{v}(\pi).$$

Proof of a: use the formula for the change of the value of R_j under a change of units, i.e. $\hat{v}(R_j) = \dots$. Use the definition of dimensionless combination and find $\hat{v}(\pi)$ inserting the expressions for $\hat{v}(R_j)$. You obtain $\hat{v}(\pi) = \text{"product"} \cdot v(\pi)$ and you can show that "product" is equal to 1.

b) $R_1, \dots, R_n$ are given physical quantities and there are $R_1, \dots, R_r$ independent ones with $r \leq \ell$, and $v(R_1) > 0, v(R_2) > 0, \dots v(R_r) > 0$, then there exist $x_i > 0$ (and a corresponding change of units) such that

$$\hat{v}(R_1) = \hat{v}(R_2) = \cdots = \hat{v}(R_r) = 1$$

Proof of b: consider $y_j := \log(v(R_j)$, $b_j := \log(\hat{v})(R_j))$ and the equation

$$\hat{v}(R_j) = \dots$$

which you used also in the previous proof.
Take logs on both sides of this equation. This will give you a linear algebraic system of equations with as right hand side a vector with components $b_j - y_j$, $j = 1, \dots, r$, and with matrix of coefficients given by "parts" of the dimension matrix A.
Figure out when this system is solvable, and how many solutions one gets. Consider in particular the case where $b_j = 0$, for $j = 1, \dots, r$. Do you get solutions in this case? These are the ones you look for.

Buckingham-Π: LEMMA 3

If $\pi_1, \ldots, \pi_{n-r}$ are independent dimensionless combinations of $R_1, \ldots, R_n$, and $r = \mathrm{rank}(A)$ then for any Φ satisfying (A3) there exists a Ψ such that

$$\Phi(R_1, \ldots, R_n) = 0 \Leftrightarrow \Psi(\pi_1, \ldots, \pi_{n-r}) = 0.$$

Proof: (STEP 1). Since $r = \mathrm{rank}(A)$ there are r independent physical quantities among $R_1, \ldots, R_n$, and without loss of generality we can assume they are $R_1, \ldots, R_r$.
Start by proving that

$$\frac{R_{r+j}}{R_1^{\lambda_{1j}} \cdots R_r^{\lambda_{rj}}} = \frac{v(R_{r+j})}{v(R_1)^{\lambda_{1j}} \cdots v(R_r)^{\lambda_{rj}}} = \tilde{\pi}_{r+j}, \quad j = 1, \ldots, n-r$$

and π_{r+j} is dimensionless. Then prove that

$$\tilde{\pi}_{r+j} = \pi_1^{\varphi_{1,j}} \cdots \pi_{n-r}^{\varphi_{n-r,j}}$$

using that $\pi_1, \ldots \pi_{n-r}$ are independent dimensionless combinations. So

$$R_{r+j} = \pi_1^{\varphi_{1,j}} \cdots \pi_{n-r}^{\varphi_{n-r,j}} R_1^{\lambda_{1j}} \cdots R_r^{\lambda_{rj}}, \qquad j = 1, \ldots, n-r.$$

(STEP 2). From the result of step 1, you see that there is a one to one correspondence

$$(R_1, \dots, R_n) \leftrightarrow (R_1, \dots, R_r, \pi_1, \dots, \pi_{n-r})$$

Define

$$\tilde{\Psi}(R_1, \dots, R_r, \pi_1, \dots, \pi_{n-r}) := \Phi(R_1, \dots, R_r, \pi_1^{\varphi_{1,1}} \cdots \pi_{n-r}^{\varphi_{n-r,1}} R_1^{\lambda_{11}} \cdots R_r^{\lambda_{r1}}, \dots)$$

Observe that $\tilde{\Psi} = 0$ is physically meaningful since $\Phi = 0$ is. So

$$\tilde{\Psi}(v(R_1), \dots, v(R_r), v(\pi_1), \dots, v(\pi_{n-r})) = 0 \Leftrightarrow \tilde{\Psi}(\hat{v}(R_1), \dots, \hat{v}(R_r), \hat{v}(\pi_1), \dots,$$

for all changes of units. From Lemma 2 there is a particular change of unit $\hat{F}_1, \dots, \hat{F}_\ell$ such that

$$\tilde{\Psi}(\hat{v}(R_1), \dots, \hat{v}(R_r), \hat{v}(\pi_1), \dots, \hat{v}(\pi_{n-r})) = \tilde{\Psi}(1, \dots, 1, v(\pi_1), \dots, v(\pi_{n-r})).$$

Define finally

$$\Psi(y_1, \dots, y_{n-r}) := \tilde{\Psi}(1, \dots, 1, y_1, \dots, y_{n-r}),$$

to conclude the proof of the lemma and of the Buckingham's Π-theorem.

Please

When you have performed all the steps, write down the complete proof and hand it in to Anders (nice handwriting is fine with us).
You will not get a mark for this, but we will use your proofs to understand how far you have been able to reach on your own and how good are your skills in linear algebra.
A complete proof following these steps will be published on the webpage.