Solution Suggestions

TMA4192 Differential Topology

NTNU, Spring 2023

All manifolds are smooth without boundary unless stated otherwise.

1. (a) Define the derivative dg_x of a smooth map $g\colon \mathbb{R}^n \to \mathbb{R}^m$ for $x \in \mathbb{R}^n$.

> **Solution:** Write in terms of a limit. $dg_x(h) = \lim_{t \mapsto 0}(g(x+th) - g(x))/t$.

Let $f\colon X \to Y$ be a smooth map between manifolds $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$.

(b) Define the derivative df_x of f at a point $x \in X$.

> **Solution:** Let $\phi\colon U \to X$ be a parameterisation around $x \in X$ such that $\phi(u) = x$, $U \subset \mathbb{R}^n$. Also let $\psi\colon V \to Y$ be a parameterisation around $f(x)$, $V \subset \mathbb{R}^m$. Define $\theta = \psi^{-1} \circ f \circ \phi\colon \mathbb{R}^n \to \mathbb{R}^m$ such that the relevant commutative diagram commutes. The derivatives of ϕ^{-1}, ψ and θ are defined using limits, so define $df_x = d\psi_{\theta(u)} \circ d\theta_u \circ d\phi_x^{-1}\colon T_xX \to T_{f(x)}Y$ making the relevant induced commutative diagram commute.

(c) What does it mean for f to be a submersion?

> **Solution:** $df_x\colon T_xX \to T_{f(x)}Y$ is surjective for every $x \in X$.

(d) State the Preimage Theorem.

> **Solution:** If y is a regular value of $f\colon X \to Y$, then $f^{-1}(y) \subset X$ is a submanifold of dimension $\dim Y - \dim X$.

Let $M(n)$ denote the space of real $(n \times n)$-matrices. Let $S(n) \subset M(n)$ be the subset of symmetric matrices A such that $A{=}A^t$, where A^t is the transpose of A. Let $O(n) \subset M(n)$ be the set of matrices that satisfy $AA^t = I$ where I is the $(n \times n)$-identity matrix.

(e) Show that the group $O(n)$ is a manifold.

> **Solution:** Let $f\colon M(n) \to S(n)$, $A \mapsto AA^t$. Then $(AA^t)^t = AA^t$, so $f(A) \in S(n)$. We want to show that I is a regular value of f. Since f can be considered as a map between Euclidean spaces, $df_A(B) = \lim_{s \mapsto 0}(f(A{+}sB) - f(A))/s = \ldots = AB^t + BA^t$. Also $T_AM(n) = M(n)$ and $T_{f(A)}S(n) = S(n)$. Suppose $f(A) = I$. We need to show that $df_A\colon M(n) \to S(n)$ is surjective. Let $C \in S(n)$. Then $df_A(B) = C$ for $B = 1/2CA$. Hence df_A is surjective. Therefore $O(n) = f^{-1}(I)$ is a manifold by the Preimage Theorem.

(20 points)

2. A manifold X is simply-connected if it is connected and every smooth map $S^1 \to X$ is homotopic to a constant map. Recall stereographic projection gives a map

$$S^n \setminus \{p\} \to \mathbb{R}^n$$

for a point $p \in S^n$. Use Sard's Theorem and the fact that $\mathbb{R}^n$ is contractible to show that S^n is simply connected for $n \geqslant 2$.

> **Solution:** Let $f\colon S^1 \to S^n$ be a smooth map. Sard's theorem implies that there is a point $p \in S^n$ such that p is a regular value of f. Let $\pi\colon S^n \setminus \{p\} \to \mathbb{R}^n$ be the stereographic projection around p.
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> If there is an $x \in S^1$ such that $p = f(x)$, then $df_x\colon T_xS^1 \to T_pS^n$ is a map from a 1-dimensional space to an n-dimensional space. This cannot be surjective for dimension reasons. Hence $p \notin \operatorname{im} f$.
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> Then $\pi \circ f\colon S^1 \to S^n \setminus \{p\} \to \mathbb{R}^n$ is nullhomotopic since $S^n \setminus \{p\} \cong \mathbb{R}^n$ and $\mathbb{R}^n$ is contractible. That is, the image of f is contractible and f is nullhomotopic.

(15 points)

3. (a) If $f\colon X \to Y$ is a smooth map of smooth manifolds, $Z \subset Y$ a submanifold, what does it mean if f is transverse to Z?

> **Solution:** At any point $x \in f^{-1}(Z)$, $\operatorname{im} df_x + T_{f(x)}Z = T_{f(x)}Y$.

(b) Let Z be the x-axis and consider a map $f\colon \mathbb{R} \to \mathbb{R}^2$ given by $f(t) = (t, t^2 - 1)$. Is f transverse to Z? Give a proof.

> **Solution:** The preimage of Z is $\{-1\}, \{1\}$. By calculating the Jacobian, $df_t = (1, 2t)\colon \mathbb{R} \to \mathbb{R}^2$. This has rank 2 at $t = \pm 1$, , so is surjective. Hence f is transverse to Z.

(c) If submanifolds X, Z in a manifold Y are transverse, show that the intersection $X \cap Z$ is also a submanifold of Y.

> **Solution:** Let $i\colon X \hookrightarrow Y$ be the inclusion map. Then $i^{-1}(Z) = X \cap Z$. Also $di_x = \mathrm{id}_x$ for all $x \in X \cap Z$, so this is always surjective. By the theorem that says transverse intersections yield submanifolds (aka an equivalent version of the Preimage Theorem), $i^{-1}(Z)$ is a submanifold of Y.

(d) For each of the following you may either give explicit expressions of spaces or give a clear sketch. Justify your examples with explanation.

i. Give an example where $\dim T_xX + \dim T_xZ = \dim T_xY$ but X and Z do not meet transversally.

Solution: Let $X = Z$ be the x-axis, $Y = \mathbb{R}^2$. Both X and Z are 1-dimensional, but the span of T_xX and T_xZ is also 1-dimensional.

ii. Give an example where X and Z do not meet transversally and $X \cap Z$ is not a submanifold of Y.

Solution: Let $Y = \mathbb{R}^2$, Z be the x-axis, and let X be a curve that intersects Z over both an interval and a point outside of the interval. The (disjoint) union of the interval and the point is not a manifold.

iii. Give an example of X, Z, Y, Y' where X is transverse to Z as submanifolds in Y but not as submanifolds in Y'.

Solution: Let $Y = \mathbb{R}^2$, $Y' = \mathbb{R}^3$, X be the x-axis, Z be the y-axis. The span of T_xX and T_xZ is equal to T_xY, a 2-dimensional subspace of T_xY'.

(13 points)

4. Recall that if y is a regular value of a smooth map $f\colon X \to Y$, then

$$T_x(f^{-1}(y)) = \ker df_x \quad \text{for any } x \in f^{-1}(y).$$

Consider a hyperboloid X and a sphere of radius $a \leqslant 1$ given by

$$X = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1\}$$
$$Z_a = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = a\}.$$

For what values of $a \leqslant 1$ do X and Z_a meet transversally in $Y = \mathbb{R}^3$? Give a proof.

Solution: Let $f\colon \mathbb{R}^3 \to \mathbb{R}$, $(x, y, z) \mapsto x^2+y^2-z^2$, and let $g\colon \mathbb{R}^3 \to \mathbb{R}$, $(x, y, z) \mapsto x^2 + y^2 + z^2$. Then $X = f^{-1}(1)$, $Z_a = g^{-1}(a)$. We need to check whether 1 and a are regular values of f and g respectively. By calculating the Jacobian, $df_{(x,y,z)} = (2x \quad 2y \quad -2z)$ and $dg_{(x,y,z)} = (2x \quad 2y \quad 2z)$. Both of these are surjective as long as $(x, y, z) \neq (0, 0, 0)$.

Since $T_{(x,y,z)}X = \ker df_{(x,y,z)}$ and $T_{(x,y,z)}Z_a = \ker dg_{(x,y,z)}$, the intersection is transverse if $\ker df_{(x,y,z)} + \ker dg_{(x,y,z)} = \mathbb{R}^3$ for every $(x, y, z) \in X \cap Z_a$. If $a < 1$, then $X \cap Z_a$ is empty. Hence X meets Z_a transversally by definition. If $a = 1$, then $X \cap Z_a$ is all points where $x^2 + y^2 - z^2 = x^2 + y^2 + z^2$, that is, when $z = 0$. Therefore $df_{(x,y,0)} = (2x \quad 2y \quad 0) = dg_{(x,y,0)}$. Both kernels are the z-axis, hence the intersection is not transverse.

(15 points)

5. (a) Suppose X is a manifold with boundary and a y is a regular value of a smooth map of manifolds $f\colon X \to Y$. According to the Preimage Theorem for manifolds with boundary, what is $\partial(f^{-1}(y))$?

Solution: $\partial(f^{-1}(y)) = f^{-1}(y) \cap \partial X$.

(b) Using the classification of one-dimensional compact manifolds and Sard's Theorem, show that if X is a compact manifold with boundary, then there is no retraction of X onto its boundary.

Solution: Suppose there is a retraction $r\colon X \to \partial X$. Then $\mathrm{id} = r \circ i\colon \partial X \to \partial X$ and $\mathrm{id} \simeq i \circ r\colon X \to X$. By Sard's Theorem there is a $y \in \partial X$ such that y is a regular value of r and of ∂r. Then $r^{-1}(y) \subset X$ is a submanifold of dimension 1 by the Preimage Theorem. It's compact since X is compact. It's also closed since $\{y\}$ is closed. Then $\partial(r^{-1}(y)) = r^{-1}(y) \cap \partial X$ has an even number of components by the classification of one-dimensional compact manifolds.

Also $(\partial r)^{-1}(y) = r^{-1}(y) \cap \partial X$ by definition of the boundary $\partial r\colon \partial X \to \partial X$ of r. Since $\partial r = \mathrm{id}$ for a retraction, $(\partial r)^{-1}(y) = \{y\}$. This contradicts the fact there should be an even number of boundary points. So there is no retraction.

(18 points)

6. Suppose X is a compact manifold, Z is a closed submanifold of a manifold Y, and that $\dim X + \dim Z = \dim Y$. Let $f\colon X \to Y$ be a smooth map that is transverse to Z.

(a) Define the mod-2 intersection number of f with Z and justify why it is defined.

Solution: $I_2(f, Z) = \#f^{-1}(Z) \mod 2$, where $f^{-1}(Z)$ is a 0-dimensional manifold for dimension reasons and has finitely many elements since it is compact.

(b) Suppose $f_0\colon X \to Y$ and $f_1\colon X \to Y$ are both transverse to Z. Also suppose there is a homotopy $F\colon X \times [0,1] \to Y$ between f_0 and f_1 such that $F \pitchfork Z$ and $\partial F \pitchfork Z$. Prove that $I_2(f_0, Z) = I_2(f_1, Z)$.

Solution: Since X is a compact manifold without boundary, $X \times [0,1]$ is a compact manifold with boundary. Then by the Preimage Theorem, $F^{-1}(Z)$ is a submanifold with boundary $\partial(F)^{-1}(Z) = F^{-1}(Z) \cap (\partial(X \times [0,1])$ and of dimension $\dim(X \times [0,1])) - (\dim Y - \dim Z)$. For dimension reasons it is a 1-dim manifold, so the boundary has an even number of elements. That is $\#\partial(F)^{-1}(Z) = \#f_0^{-1}(Z) + \#f_1^{-1}(Z)$ is even. So $\#f_0^{-1}(Z) \equiv \#f_1^{-1}(Z)$ mod 2.

(c) For each of the following give reasons for your answers.

i. Consider $Y = \mathbb{R}P^2$ as the quotient of $S^2 \subset \mathbb{R}^3$ under the antipodal map. Let $X = \{[x_0 : x_1 : x_2] \in \mathbb{R}P^2 \mid x_0 = x_1\}$. What is $I_2(X, X)$ and why is it defined?

> **Solution:** Deform one copy of X a little to get a line that intersects the other copy of X at an odd number of points. Then $I_2(X, X) = 1$.

ii. Let $Y = T^3 = S^1 \times S^1 \times S^1$ be a 3-dimensional torus. Give an example of subspaces X and Z in Y such that $I_2(X, Z) \neq 0$.

> **Solution:** Just like $T^2 = S^1 \times S^1$ can be obtained by identifying opposite sides of a square, T^3 is obtained by identifying opposite sides of a cube. Let $X = S^1 \times S^1 \times \{1\}$, $Z = \{1\} \times \{1\} \times S^1$. Inside the cube, X is a plane and Z is a line perpendicular to the plane. They intersect at one point.

(19 points)