

TMA4192 Differential Topology
NTNU, Spring 2023

All manifolds are smooth without boundary unless stated otherwise.

1. (a) Define the derivative dg_x of a smooth map $g: \mathbb{R}^n \rightarrow \mathbb{R}^m$ for $x \in \mathbb{R}^n$.

Let $f: X \rightarrow Y$ be a smooth map between manifolds $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$.

- (b) Define the derivative df_x of f at a point $x \in X$.
(c) What does it mean for f to be a submersion?
(d) State the Preimage Theorem.

Let $M(n)$ denote the space of real $(n \times n)$ -matrices. Let $S(n) \subset M(n)$ be the subset of symmetric matrices A such that $A=A^t$, where A^t is the transpose of A . Let $O(n) \subset M(n)$ be the set of matrices that satisfy $AA^t = I$ where I is the $(n \times n)$ -identity matrix.

- (e) Show that the group $O(n)$ is a manifold.

(20 points)

2. A manifold X is simply-connected if it is connected and every smooth map $S^1 \rightarrow X$ is homotopic to a constant map. Recall stereographic projection gives a map

$$S^n \setminus \{p\} \rightarrow \mathbb{R}^n$$

for a point $p \in S^n$. Use Sard's Theorem and the fact that \mathbb{R}^n is contractible to show that S^n is simply connected for $n \geq 2$.

(15 points)

3. (a) If $f: X \rightarrow Y$ is a smooth map of smooth manifolds, $Z \subset Y$ a submanifold, what does it mean if f is transverse to Z ?
(b) Let Z be the x -axis and consider a map $f: \mathbb{R} \rightarrow \mathbb{R}^2$ given by $f(t) = (t, t^2 - 1)$. Is f transverse to Z ? Give a proof.
(c) If submanifolds X, Z in a manifold Y are transverse, show that the intersection $X \cap Z$ is also a submanifold of Y .
(d) For each of the following you may either give explicit expressions of spaces or give a clear sketch. Justify your examples with explanation.
i. Give an example where $\dim T_x X + \dim T_x Z = \dim T_x Y$ but X and Z do not meet transversally.
ii. Give an example where X and Z do not meet transversally and $X \cap Z$ is not a submanifold of Y .
iii. Give an example of X, Z, Y, Y' where X is transverse to Z as submanifolds in Y but not as submanifolds in Y' .

(13 points)

4. Recall that if y is a regular value of a smooth map $f: X \rightarrow Y$, then

$$T_x(f^{-1}(y)) = \ker df_x \quad \text{for any } x \in f^{-1}(y).$$

Consider a hyperboloid X and a sphere of radius $a \leq 1$ given by

$$\begin{aligned} X &= \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1\} \\ Z_a &= \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = a\}. \end{aligned}$$

For what values of $a \leq 1$ do X and Z_a meet transversally in $Y = \mathbb{R}^3$?
Give a proof. **(15 points)**

5. (a) Suppose X is a manifold with boundary and y is a regular value of a smooth map of manifolds $f: X \rightarrow Y$. According to the Preimage Theorem for manifolds with boundary, what is $\partial(f^{-1}(y))$?
- (b) Using the classification of one-dimensional compact manifolds and Sard's Theorem, show that if X is a compact manifold with boundary, then there is no retraction of X onto its boundary.

(18 points)

6. Suppose X is a compact manifold, Z is a closed submanifold of a manifold Y , and that $\dim X + \dim Z = \dim Y$. Let $f: X \rightarrow Y$ be a smooth map that is transverse to Z .

- (a) Define the mod-2 intersection number of f with Z and justify why it is defined.
- (b) Suppose $f_0: X \rightarrow Y$ and $f_1: X \rightarrow Y$ are both transverse to Z . Also suppose there is a homotopy $F: X \times [0, 1] \rightarrow Y$ between f_0 and f_1 such that $F \pitchfork Z$ and $\partial F \pitchfork Z$. Prove that $I_2(f_0, Z) = I_2(f_1, Z)$.
- (c) For each of the following give reasons for your answers.
- i. Consider $Y = \mathbb{R}P^2$ as the quotient of $S^2 \subset \mathbb{R}^3$ under the antipodal map. Let $X = \{[x_0: x_1: x_2] \in \mathbb{R}P^2 \mid x_0 = x_1\}$. What is $I_2(X, X)$ and why is it defined?
 - ii. Let $Y = T^3 = S^1 \times S^1 \times S^1$ be a 3-dimensional torus. Give an example of subspaces X and Z in Y such that $I_2(X, Z) \neq 0$.

(19 points)