Exam document

TMA4192 Differential Topology

NTNU, Spring 2023

Feedback

The standard NTNU grading scale was used when assigning grades:

- A: 89–100 points
- B: 77–88 points
- C: 65–76 points
- D: 53–64 points
- E: 41–52 points
- F: 0–40 points

1. (a) (1 point) Define the derivative dg_x of a smooth map $g\colon \mathbb{R}^n \to \mathbb{R}^m$ for $x \in \mathbb{R}^n$.

   Let $f\colon X \to Y$ be a smooth map between manifolds $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$.

   (b) (6 points) Define the derivative df_x of f at a point $x \in X$.

   (c) (1 point) What does it mean for f to be a submersion?

   (d) (2 points) State the Preimage Theorem.

   Let $M(n)$ denote the space of real $(n \times n)$-matrices. Let $S(n) \subset M(n)$ be the subset of symmetric matrices A such that $A{=}A^t$, where A^t is the transpose of A. Let $O(n) \subset M(n)$ be the set of matrices that satisfy $AA^t = I$ where I is the $(n \times n)$-identity matrix.

   (e) (10 points) Show that the group $O(n)$ is a manifold.

   Feedback

   On the whole, students did very well on this question.

2. (15 points) A manifold X is simply-connected if it is connected and every smooth map $S^1 \to X$ is homotopic to a constant map. Recall stereographic projection gives a map

$$S^n \setminus \{p\} \to \mathbb{R}^n$$

   for a point $p \in S^n$. Use Sard's Theorem and the fact that $\mathbb{R}^n$ is contractible to show that S^n is simply connected for $n \geqslant 2$.

   Feedback

   Most students did well on this question, although some solutions were a little unclear.

3. (a) (2 points) If $f\colon X \to Y$ is a smooth map of smooth manifolds, $Z \subset Y$ a submanifold, what does it mean if f is transverse to Z?

   (b) (4 points) Let Z be the x-axis and consider a map $f\colon \mathbb{R} \to \mathbb{R}^2$ given by $f(t) = (t, t^2 - 1)$. Is f transverse to Z? Give a proof.

   (c) (4 points) If submanifolds X, Z in a manifold Y are transverse, show that the intersection $X \cap Z$ is also a submanifold of Y.

   (d) (3 points) For each of the following you may either give explicit expressions of spaces or give a clear sketch. Justify your examples with explanation.

      i. Give an example where $\dim T_x X + \dim T_x Z = \dim T_x Y$ but X and Z do not meet transversally.

      ii. Give an example where X and Z do not meet transversally and $X \cap Z$ is not a submanifold of Y.

      iii. Give an example of X, Z, Y, Y' where X is transverse to Z as submanifolds in Y but not as submanifolds in Y'.

Feedback

Almost everyone did very well in Parts (a), (b), (d.i) and (d.iii). Part (c) was difficult for some, and most students skipped part (d.ii).

4. (15 points) Recall that if y is a regular value of a smooth map $f\colon X \to Y$, then

$$T_x(f^{-1}(y)) = \ker df_x \quad \text{for any } x \in f^{-1}(y).$$

Consider a hyperboloid X and a sphere of radius $a \leqslant 1$ given by

$$X = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1\}$$
$$Z_a = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = a\}.$$

For what values of $a \leqslant 1$ do X and Z_a meet transversally in $Y = \mathbb{R}^3$?
Give a proof.

Feedback

Mostly answered well. Some solutions were a little unclear.

5. (a) (2 points) Suppose X is a manifold with boundary and a y is a regular value of a smooth map of manifolds $f\colon X \to Y$. According to the Preimage Theorem for manifolds with boundary, what is $\partial(f^{-1}(y))$?

(b) (16 points) Using the classification of one-dimensional compact manifolds and Sard's Theorem, show that if X is a compact manifold with boundary, then there is no retraction of X onto its boundary.

Feedback

Part (a) was straight-forward for many. Part (b) was very split: on the whole students either answered this very well or struggled with all parts. The main idea is to suppose there is a retraction. We use Sard's Theorem to pick a regular value, then the Preimage Theorem let's us think about the preimage of this regular value. We get a contradiction when looking at the number of boundary points of the preimage.

6. Suppose X is a compact manifold, Z is a closed submanifold of a manifold Y, and that $\dim X + \dim Z = \dim Y$. Let $f\colon X \to Y$ be a smooth map that is transverse to Z.

   (a) (3 points) Define the mod-2 intersection number of f with Z and justify why it is defined.

   (b) (16 points) Suppose $f_0\colon X \to Y$ and $f_1\colon X \to Y$ are both transverse to Z. Also suppose there is a homotopy $F\colon X \times [0,1] \to Y$ between f_0 and f_1 such that $F \pitchfork Z$ and $\partial F \pitchfork Z$. Prove that $I_2(f_0, Z) = I_2(f_1, Z)$.

   (c) For each of the following give reasons for your answers.

      i. (3 points) Consider $Y = \mathbb{R}P^2$ as the quotient of $S^2 \subset \mathbb{R}^3$ under the antipodal map. Let $X = \{[x_0\colon x_1\colon x_2] \in \mathbb{R}P^2 \mid x_0 = x_1\}$. What is $I_2(X, X)$ and why is it defined?

      ii. (5 points) Let $Y = T^3 = S^1 \times S^1 \times S^1$ be a 3-dimensional torus. Give an example of subspaces X and Z in Y such that $I_2(X, Z) \neq 0$.

Feedback

Part (a) was mostly well done. Part (b) was very split: mostly students either answered it very well or struggled/skipped the question. Some students attempted Part (c.i). Not many students attempted Part (c.ii).

In general there were very few pictures drawn for Part (c) even though it would definitively have help. Instead, arguments were generally fairly analytical. The idea with Part (c.i) was that this example is equivalent to an example from lectures where we had the self-intersection of the central line on the square model of the Möbius strip. In this exam question we had the self-intersection of a diagonal line on the square model of $\mathbb{R}P^2$. In lectures we deformed one of the copies of the line and used something like the Intermediate Value Theorem to argue why there would be an odd number of intersections.

Part (c.ii) was also equivalent to a 2-dimensional torus example from lectures. In this 3-dimensional version, one technique was to think about the intersection of a plane and a line inside a cube. A common mistake in this part was not checking dimensions, for example suggesting $X = S^1 \times \{1\} \times \{1\}$ and $Z = \{1\} \times S^1 \times \{1\}$, so that $\dim X + \dim Z = 2 \neq \dim Y = 3$.