TMA4190: Introduction to Topology.

Final Exam

- Let $n \geq 1$. We will always denote by $\mathbb{R}^n$ the set consisting of n-tuples of real numbers equipped with the standard topology (the one you know from Calculus).

- We denote by $I \subset \mathbb{R}$ the unit interval $I = \{x \in \mathbb{R}|\ 0 \leq x \leq 1\}$ equipped with the subspace topology.

Problem 1. *Define the notion of topological space and continuous map between topological spaces and provide the following examples:*

i) Define the discrete topology on a set X and show that if Y is another topological space then any function $f\colon X \to Y$ is continuous.

ii) Define the indiscrete topology on a set Y and show that if X is another topological space then every function $f\colon X \to Y$ is continuous.

Solution. A topological space is a pair (X, τ) where X is a set and τ is a collection of subsets of X called the open sets. We require τ to satisfy the following properties:

- We have $\emptyset, X \in \tau$.

- An arbitrary union of elements $U_i \in \tau$ for $i \in I$ remains in τ.

- A finite intersection of elements $U_j \in \tau$ for $j \in J$ and $|J| < \infty$ remains in τ.

A map $f\colon X \to Y$ between topological spaces is said to be continuous if for every open set $U \subseteq Y$ we have that $f^{-1}(U)$ is open in X.

i) The discrete topology is defined by declaring every subset of X to be open. Given any map $f\colon X \to Y$ then it follows that for any open set $U \subseteq Y$ then $f^{-1}(U)$ is a subset of X and hence open.

ii) The indescrete topology on a set Y has as open sets $\{\emptyset, Y\}$. Given $f\colon X \to Y$ we note that $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$ which shows that f is continuous.

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Problem 2. *Let $A \subset X$ where X is a topological space and such that $A \neq \emptyset$.*

i) Define A° the interior of A, and $\overline{A}$ the closure of A.

ii) Let A^c denote the complementary of A in X. Show that $\overline{A} = X$ if and only if $(A^c)^{\circ} = \emptyset$.

iii) What do we call a subset satisfying the equivalent conditions above? Give an example (without proof) of such $A \subset X$ with the property above.

Solution. i) The interior of A, is the union of all open sets U which are contained in A. The closure of A is the intersection of all closed sets which contain A.

ii) We have that $x \in (A^c)^{\circ}$ if and only if there exists some open set U, such that $U \subseteq A^c$. This is equivalent to the condition that $U \cap A = \emptyset$. This means precisely that $x \notin \overline{A}$. We conclude that

$$(A^c)^{\circ} = (\overline{A})^c.$$

The result follows from the equality above.

iii) We call such subsets dense. In the lecture, we saw that $\mathbb{Q} \subset \mathbb{R}$ is dense. ■

Problem 3. *Let $A \subset X$ where X is a topological space and $A \neq \emptyset$.*

i) Define the subspace topology on the subset A.

ii) Show that a subset $Z \subseteq A$ is closed in the subspace topology if and only if there exists some $Z_X \subseteq X$ which is closed in X such that $Z_X \cap A = Z$.

Solution. i) The subspace topology on A consists in subsets of the form $U \cap A$ where U is open in X.

ii) Suppose that $Z \subseteq A$ is closed in A. Then there exists an open $U \subseteq X$ of X such that $U \cap A = A \setminus Z$. Let $Z_X = X \setminus U$ (which is closed by definition). We check that

$$(X \setminus U) \cap A = A \setminus (U \cap A) = A \setminus (A \setminus Z) = Z.$$

For the other direction we consider a closed subset in $L \subseteq X$ and set $L \cap A = Z$. We compute

$$(X \setminus L) \cap A = A \setminus (L \cap A) = A \setminus Z.$$

Since $X \setminus L$ is open by definition we conclude that Z is closed in the subspace topology.

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Problem 4. *Let X, Y be topological spaces.*

i) Define the product topology on the cartesian product $X \times Y$.

ii) Show that given a pair of continuous maps $f_X : A \to X$ and $f_Y : A \to Y$ then there exists a unique continuous map $f : A \to X \times Y$ such that $f_X = \pi_X \circ f$ and $f_Y = \pi_Y \circ f$ where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are the canonical projection maps.

Solution. i) The product topology is generated by the basis elements $U \times V$ where $U \subseteq X$ is open in X and $V \subseteq Y$ is open in Y.

ii) We define $f : A \to X \times Y$ as $f(a) = (f_X(a), f_Y(a))$. Note that this choice of f is unique by hypothesis so the only thing that we need to show that f is continuous. As we saw in the theory, it will suffice to show that $f^{-1}(W)$ is open where W is an element of the basis.

Observe that if $W = U \times V$ it follows that $W = \pi_X^{-1}(U) \cap \pi_Y^{-1}(V)$ since $\pi_X^{-1}(U) = U \times Y$ and $\pi_Y^{-1}(V) = X \times V$. We conclude that

$$f^{-1}(W) = f^{-1}\left(\pi_X^{-1}(U) \cap \pi_Y^{-1}(V)\right) = f_X^{-1}(U) \cap f_Y^{-1}(V)$$

which is open since f_X and f_Y are continuous by assumption.

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Problem 5. *Let X, Y, Z be a topological spaces.*

i) *What does it mean for X to satisfy the Hausdorff property?*

ii) *Let $A \subset Y$ such that $\overline{A} = Y$ and consider a pair of continuous maps $f, g \colon Y \to Z$ such that $f(a) = g(a)$ for every $a \in A$. Show that if Z is Hausdorff then $f = g$.*

Solution. i) X is said to satisfy the Hausdorff property if given $x \neq y$ points in X, then there exists open neighbourhoods U, V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

ii) Let us suppose for contradiction that there exists a point $\alpha \in Y$ such that $f(\alpha) \neq g(\alpha)$. We pick disjoint neighbourhoods $U_{f(\alpha)}$ and $V_{g(\alpha)}$ and observe that

$$\alpha \in f^{-1}(U_{f(\alpha)}) \cap g^{-1}(V_{g(\alpha)}) = W.$$

Since f, g are continouous it follows that W is open and by density of A we obtain that $W \cap A \neq \emptyset$. Let $a \in W \cap A$, then it follows that $f(a) = g(a) \in U_{f(\alpha)} \cap U_{g(\alpha)} = \emptyset$ which is a contradiction.

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Problem 6. *Let X be a topological space.*

i) *Define what it means for X to be compact.*

ii) *Show that $\mathbb{R}^n$* is not *compact for $n \geq 1$.*

iii) *Use ii) to show that $I^n = I \times I \times \cdots \times I$ (n-dimensional cube) is not homeomorphic to $\mathbb{R}^n$.*

Solution. i) A space is said to be compact if given a covering by open sets $\{U_i\}_{i \in I}$, where $\bigcup_{i \in I} U_i = X$ then there exists a finite subcollection $\{i_0, \ldots, i_n\}$ such that

$$\bigcup_{j=0}^{n} U_{i_j} = X.$$

In other words, every open cover admits a finite refinement.

ii) We consider a cover by open balls $\{B(0, n)\}_{n \geq 1}$. Then if this cover were finite it would follow that $\mathbb{R}^n \subseteq B(0, M)$ for some $M > 0$. This is a contradiction since in $\mathbb{R}^n$ points can be at arbitrary long distance from one another.

iii) From the theory we know that I is compact and that a finite product of compact spaces is again compact. Since compactness is a topological property we conclude that I^n is not homeomorphic from $\mathbb{R}^n$ since once space is compact and the other is not.

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Problem 7. *Let X be a topological space.*

i) Define what it means for X to be connected.

ii) Show that given a continuous map $f\colon A \to X$ such that A is connected then the image $f(A) \subseteq X$ is connected.

Solution. i) A space X is connected if we cannot find open subsets U, V such that $U, V \neq \emptyset$ and $U, V \neq X$, and such that $U \cap V = \emptyset$ and $U \cup V = X$. A pair U, V as above is called a separation.

ii) Suppose that we have a separation U, V of $f(A)$. We claim that $f^{-1}(U), f^{-1}(V)$ is a separation of A. First we observe that if $f^{-1}(U) = A$ then it follows that $f(A) \subseteq U$ and so $V = \emptyset$ and similarly if $f^{-1}(U) = \emptyset$. We also note that

$$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = \emptyset,$$

$$f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V) = f^{-1}(f(A)) = A.$$

We conclude that we have produced a separation which contradicts our hypothesis.

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Problem 8. *Let $f, g\colon X \to Y$ be continuous maps of topological spaces.*

i) What do we mean when we say that f is homotopic to g?

ii) Show that if we equip Y with the indiscrete topology then f is homotopic to g.

iii) Define what it means for a continuous map $f : X \to Y$ to be a homotopy equivalence.

iv) Show that any topological space equipped with the indiscrete topology must be homotopy equivalent to a point.

Solution. i) We say that f, g is homotopic to g if there exists a continuous map $H\colon X \times I \to Y$ such that $H(-,0) = f$ and $H(-,1) = g$.

ii) We know that every map with target a space with the indiscrete topology is continuous. We can construct a homotopy as follows

$$H(x,t) = \begin{cases} f(x), & \text{if } x = 0 \\ g(x), & \text{if } x \neq 0. \end{cases}$$

iii) A map $f\colon X \to Y$ is a homotopy equivalence if and only if there exists another map $g\colon Y \to X$ such that $g \circ f$ is homotopic to the identity on X and $f \circ g$ is homotopic to the identity map on Y.

iv) Let X be a space with the indiscrete topology and consider the unique map to the point $t : X \to *$ and let $i : * \to X$ be any map selecting a point in X. Then clearly $t \circ i$ is the identity on $*$. Finally we look at $i \circ t$ and use *ii*) to see that $i \circ t$ must be homotopic to the identity map on X.

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Problem 9. *Let $T = \{$Hausdorff, compact, connected, path connected$\}$ be the set of topological properties which we have studied in this course. We say that $P \in T$ is invariant under homotopy equivalence if:*

- *Given a homotopy equivalence $f\colon X \to Y$ then X satisfies P if and only if Y does.*

Provide (and justify) the following examples:

i) A property of topological spaces which is invariant under homotopy equivalence.

ii) A property of topological spaces which is not invariant under homotopy equivalence.

Solution. i) Let us consider path connectedness. We show that if X is path connected so is Y (the reverse is totally analogous). Let $g\colon Y \to X$ be the inverse (up to homotopy) of f. Given $y_0, y_1 \in Y$ we pick a path from $g(y_0)$ to $g(y_1)$ in X. Applying f we obtain a path from $fg(y_0)$ to $fg(y_1)$. Finally we use the homotopy between fg and the identity on Y to obtain paths between y_i abd $fg(y_i)$ for $i = 0, 1$. The result follow by concatenation of paths.

ii) We show that being Hausdorff is not invariant under homotopy equivalence. Indeed, a indiscrete space with more than 2 points cannot be Hausdorff but we showed in the previous exercise that every indiscrete space is equivalent to a point which is clearly Hausdorff.

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Problem 10. *Let* $I = [0,1]$ *be the closed interval and define* $L = [0,1] \times [0,1]/\sim$ *where the equivalence relation identifies* $(0,y) \sim (1,y)$. *Show that* L *is path connected and compute the fundamental group of* L.

Solution. Observe that $I \times I$ is path connected and that the quotient map $I \times I \to L$ is continuous and surjective. In particular, this means given $x, y \in L$ we can pick a preimage $\hat{x}, \hat{y} \in I \times I$. Since $I \times I$ is path connected we can produce a path between $\hat{x}$ and $\hat{y}$ which becomes a path between x and y after applying the quotient map.

For the second part we show that we have a homeomorphism $L \simeq \mathbb{S}^1 \times I$. Once we have this we can use that the fundamental group of a product is the product of the fundamental groups to see that

$$\pi_1(L, p) = \pi_1(\mathbb{S}^1, a) \times \pi_1(I, b) \simeq \mathbb{Z}$$

in the last step we are using that I is contractible so it has trivial fundamental group together with the fact that the fundamental group of the circle is given by the integers.

We construct a map

$$F \colon I \times I \to \mathbb{S}^1 \times I, (s,t) \mapsto (\cos(2\pi t), \sin(2\pi t), s).$$

Observe that F can be expressed as the product of two continuous maps and is therefore continuous. The map F clearly factors through the equivalence relation defining L, and therefore the universal property of the quotient yields a map

$$\hat{F} : L \to \mathbb{S}^1 \times I.$$

By construction the map $\hat{F}$ is bijective so the only thing left to show is that $\hat{F}$ is a homeomorphism. To finish the proof, we will show that $\hat{F}$ is closed and therefore, a homeomorphism.

This follows from the fact that L is compact (quotient of a compact space) and the fact that $\mathbb{S}^1 \times I$ is Hausdorff (product of Hausdorff spaces). ■