TMA4190: Introduction to Topology.

Mock exam

Problem 1. *Let X be a topological space and let $K \subset X$ be a compact subspace such that $K \neq X$. Assume that for every pair of points $x, y \in X$ there exists a continuous map $f : X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$.*

i) Show that X is Hausdorff.

ii) Let $x \notin K$ show that there exists a continuous map $f : X \to [0,1]$ such that $f(x) = 1$ and $f(K) \subseteq [0, 1/2)$.

Solution. i) Given two different points $x, y \in X$ we pick a continuous map $f : X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$. We define $U_x = f^{-1}([0, 1/2))$ and $U_y = f^{-1}((1/2, 1])$ which are preimages of open sets under a continuous maps and are therefore open. It follows that $U_x \cap U_y = \emptyset$ and thus X is Hausdorff.

ii) Let $x \notin K$. For every $y \in K$ we define $f_y : X \to [0,1]$ to be a continuous function such that $f_y(y) = 0$ and $f_y(x) = 1$. We further define $W_y = f_y^{-1}([0, 1/2))$ which is an open set such that $y \in W_y$. We repeat this process for each $y \in K$ which provides us with an open cover

$$K \subseteq \bigcup_{y \in K} W_y.$$

Since K is compact there exists finitely many points $\{y_0, \ldots, y_n\}$ such that $K \subseteq \bigcup_{i=0}^{n} W_{y_i}$. We define a continuous function $f : X \to [0,1]$ given by the pointwise product

$$f = \prod_{i=0}^{n} f_{y_i}$$

which is well defined and continuous since we are only considering finitely many continuous functions in the definition. To finish the proof let $z \in K$. Then $z \in W_{y_j}$ for some index $0 \leq j \leq n$. Then it follows that

$$f(z) = f_j(z) \prod_{i \neq j} f_{y_i}(z) < \frac{1}{2} \prod_{i \neq j} f_{y_i}(z) \leq \frac{1}{2}.$$

■

Problem 2. *Let define $\mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\} \big/ \sim$ where the equivalence relation identifies vectors $\lambda \vec{x} \sim \vec{x}$ where $\lambda \in \mathbb{R}$ and $\lambda \neq 0$.*

i) Show that $\mathbb{R}P^n$ is path connected.

ii) Show that $\mathbb{R}P^n$ is Hausdorff.

iii) We observe that we have a map $\pi : \mathbb{S}^n \to \mathbb{R}P^n$ which identifies a pair of points $x, y \in \mathbb{S}^n$ if $x = -y$. Show that p is a covering map.

iv) Show that $\mathbb{R}P^n$ is compact.

v) Assume that we know that for $n \geq 2$ and a point $x \in \mathbb{S}^n$ we have $\pi_1(\mathbb{S}^n, x) = 0$. Compute $\pi_1(\mathbb{R}P^n, p(x))$.

Solution. Let $p : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ be the quotient map.

i) Let $x, y \in \mathbb{R}P^n$ and pick a pair of points in $a, b \in \mathbb{R}^{n+1} \setminus \{0\}$ such that $p(a) = x$ and $p(b) = y$. Then since $\mathbb{R}^{n+1} \setminus \{0\}$ is path connected we can pick a path $\gamma : I \to \mathbb{R}^{n+1} \setminus \{0\}$ joining a and b. Then $p \circ \gamma$ is the desired path between x and y.

ii) Observe that two points $x, y \in \mathbb{R}^{n+1} \setminus \{0\}$ are identified in $\mathbb{R}P^n$ if and only if there exists a straight line $L \subset \mathbb{R}^{n+1}$ passing through the origing such that $x, y \in L$.

Let us consider a pair of different points $p(x) \neq p(y)$ in $\mathbb{R}P^n$. Then by the previous discussion it follows that x, y do not belong to the same straight line passing through the origin. Let L_x be the straight line connecting the origin and x and similarly let us defined L_y. It then follow that we can find $\epsilon_x > 0$ and $\epsilon_y > 0$ such that for every $z \in B(x, \epsilon_x)$ we have that $B(y, \epsilon_y) \cap L_z = \emptyset$. This follows that after applying p we have

$$V_{p(x)} \cap V_{p(y)} = \emptyset$$

where $V_{p(x)} = p(B(x, \epsilon_x))$ and $V_{p(y)} = p(B(y, \epsilon_y))$. To finish the proof we need to show that $V_{p(x)}$ (resp. $V_{p(y)}$) is open. To see this we simply compute $p^{-1}(V_{p(x)})$ and check that it is open which is precisely the definition of open set in the quotient topology.

iii) We consider the obvious inclusion $i : \mathbb{S}^n \to \mathbb{R}^{n+1} \setminus \{0\}$ and observe that $\pi = p \circ i$. In particular, we learn that π is surjective and continuous. Let $x \in \mathbb{R}P^n$ and pick a preimage $a \in \mathbb{S}^n$ such that $\pi(a) = x$. Then it is clear that we can find a small open set U_a such that for every $z \in U_a$ it follows that $-z \notin U_a$. Then defining $U_{-a} = \{z \in \mathbb{S}^n | \ -z \in U_a\}$ one observes that
$$U_a \coprod U_{-a} = \pi^{-1}(\pi(U_a)).$$
To finish the proof one simply checks that $p^{-1}(\pi(U_a))$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$.

iv) The map π is continuous and surjective. Since $\mathbb{S}^n$ is compact it follows that $\mathbb{R}P^n$ must also be compact.

v) We know that $\mathbb{S}^n$ is simply connected for $n \geq 2$. Then the theory shows that $\pi^{-1}(x) \simeq \pi_1(\mathbb{R}P^n, x)$. However $\pi^{-1}(x)$ has precisely 2 elements and there is only one group of order 2 so we conclude $\pi_1(\mathbb{R}P^n, x) = \mathbb{Z}_2$.

■

Problem 3. *Let $p : E \to B$ be a covering map and assume that B is Hausdorff. Show that E must also be Hausdorff.*

Proof. Let $x, y \in E$ such that $x \neq y$. We consider two cases given by $p(x) \neq p(y)$ or $p(x) = p(y)$. The first case follows immediately from the fact that B is Hausdorff. Let us suppose that $p(x) = p(y) = z$ and let $z \in U \subset B$ such that
$$p^{-1}(U) \simeq \coprod_{\lambda \in \Lambda} V_\lambda.$$
Let λ_x and λ_y be the indices such that $x \in V_{\lambda_x}$ and $y \in V_{\lambda_y}$. If $\lambda_x \neq \lambda_y$ then the claim follows from the fact that $V_{\lambda_x} \cap V_{\lambda_y} = \emptyset$. Suppose that $\lambda_x = \lambda_y$ then since the restricted mpa
$$p_{|U_{\lambda_x}} : V_{\lambda_x} \to U$$
is an homeomorphism it follows that $p(x) \neq p(y)$ which is a contradiction. □

Problem 4. *Let $p : E \to B$ be a covering map and let $e_0 \in E$ such that $p(e_0) = b_0$.*

i) Show that $\pi_1(E, e_0) \to \pi_1(B, b_0)$ is a monomorphism whose image consists in those loops in B (with basepoint b_0) that admit a lift to a loop in E (with basepoint e_0).

Solution. Let us show that the map $\pi_1(E, e_0) \to \pi_1(B, b_0)$ is a monomorphism. Suppose that we are given two loops $\gamma_1, \gamma_2 : I \to E$ such that $p \circ \gamma_1 \sim p \circ \gamma_2$ in B via a homotopy $H : I \times I \to B$. Using the homotopy lifting property we obtain a homotopy $\hat{H} : I \times I \to E$ such that $p \circ \hat{H} = H$. We saw in the theory lecture that uniqueness of our lifts imply that since H is a path homotopy then so is $\hat{H}$. We observe the following:

*) $\hat{H}(-, 0)$ is a lift of $p \circ \gamma_1$ and $\hat{H}(-, 1)$ is a lift of $p \circ \gamma_2$. We can now use the uniqueness of the path lifting property to see that $\hat{H}(-, 0) = \gamma_1$ and $\hat{H}(-, 1) = \gamma_2$.

We conclude that γ_1 and γ_2 are path homotopic and thus the map $\pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective.

To finish the proof let us characterize the image of the map. Let $\tau : I \to E$ and $\gamma : I \to B$ be loops such that $p \circ \tau$ is path homotopic to γ. We consider a lift of this path homotopy to $\hat{H} : I \times I \to E$. A similar argument as before shows that $\hat{H}(-, 0) = \tau$ and that $\hat{H}(-, 1) = \hat{\gamma}$ is a lift of γ. It is only left to show that $\hat{\gamma}$ is a loop. This follows from the fact that τ is a loop and that $\hat{H}$ is a path homotopy. ■